Ambiguous Networks

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Abstract

We model the formation of cross-holding network structures. Each agent can invest in an individual primitive asset or hold shares of others whose value is conditional on their respective investment decisions. When an agent holds shares of many distinct and independent players, she is able to diversify her asset portfolio risk. However, in a chain of cross-exposures, she could indirectly find herself exposed to third agents whose expected payoff is perceived as uncertain in the sense of Knightian-uncertainty. As a direct consequence of this, the optimal holdings and cross-exposure level of each agent will be in part conditional on her risk and ambiguity aversion attitude. We characterize the optimal and the stable network structures which arise in this environment; respectively, these are the structures which maximize a utilitarian welfare function and the ones which emerge endogenously. Sparse networks are optimal and stable when the cost per-link is high and players are more sensitive to diversification than uncertainty, while the opposite is true for dense networks. In general, we find that the architecture of the stable networks is similar to the optimal ones but not necessarily of the same density; individual agents fail to internalize the positive/negative externality due to the activation of a link with another peer. The model proposes a new alternative interpretation for some of the structural changes observed in financial markets pre and post-crisis.

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1 Introduction

Financial markets are increasingly interconnected. By means of liberalization of capital flows, financial institutions are directly or indirectly exposed to global markets as never before. Prior the recent Global Financial Crisis (GFC), consensus was that higher global integration should yield greater financial stability due to increased individual risk-diversification by larger risk-pooling. The intuition is clear: when firms are more interconnected, they should be able to absorb better idiosyncratic shocks of other peers in the system. As consequence, financial networks should be more resilient to defaults of one or a few firms if the rest of the institutions are substantially exposed to many counterparties. In the aftermath of the GFC, it is evident that this was far from being the case, and to understand “what we have missed” became a priority within the agendas of the regulators and researchers across the globe (Haldane et al. [2009a]).

A first conjecture may challenge the idea that, prior to the GFC, either the global financial market was indeed diversified or that initial shocks were relatively small. However, empirical evidence seems to confirm both these points (Bartram et al. [2007], Elsinger et al. [2006], Boss et al. [2004], Furfine [2003]). Some recent theoretic contributions advanced the idea that systemic risk may be non-monotonic with respect to the network density\footnote{See for instance Battiston et al. [2012b], Stiglitz [2010], Castiglionesi & Navarro [2008], Allen & Babus [2009], Wagner [2010], Elliott et al. [2014].}; a relatively interconnected system absorbs shocks better than a less interconnected one as long as the size of the initial shock is smaller than a threshold level. Above this level, denser networks amplify the risk of contagion in the system. Most of these works study the propagation of risk due to default-cascades for a given (fixed) cross-holding network structure and an initial shock (e.g. default) of one or more institutions. One of the many interesting features of this approach is to characterize the systemic risk as function of only a few factors such as the size of the initial shock (how many firms originally default), and the number of links owned by each agent (degree). This is necessary to make the problem mathematically tractable and thus useful for policy makers who aim to quantify systemic risk and target the most systemically important institutions. However, while these models may effectively show that an integrated system is not always synonymous with robust, as previously pointed out, in order to trigger a large default-cascade, a relatively large initial shock is usually necessary.

Relatively little attention has been focused on the behavioural aspect of the problem; do individual firms change their exposure to counterparties in response to an initial shock in the system? How do firms form beliefs about risk of contagion given the complexity of the
system and its potentially dynamic nature? We argue that understanding how agents react to shocks and how they form beliefs about the real contagion risk they are facing may help to predict changes of the network over time and thus explain why a (potentially) unharmful idiosyncratic shock can still have a large impact across the system.

We propose a model in which we relate risk and uncertainty, as perceived by individual firms, to the network architecture interconnecting them. The network captures the risk-exposure between firms (the cross-holding network structure). Each firm can invest in their own primitive assets or in shares of other peers’ portfolios; the existence of a directed link from agent $i$ to agent $j$ indicates that $i$ is investing in $j$’s portfolio (but not necessarily vice versa). Each agent cares about the exposures faced by his counterparty. However, he may not know exactly the type of exposure that his counterparty faces, and experiences Knightian uncertainty (ambiguity) about this. A direct link from agent $i$ to agent $j$ gives access to perfect information about the distribution of the returns of $j$’s primitive asset while $j$’s indirect exposures to third agents’ assets could create uncertainty over the correct probability distribution to consider. If agent $i$ invests in agent $j$’s portfolio and $j$ in a third agent $k$ to which $i$ has no direct exposure, then $i$ is assumed to have incomplete information on the returns of $j$’s shares invested in $k$’s portfolio.

We start by studying the portfolio allocation problem solved by each decision maker (DM) given constrained investment possibilities. In other words, we fix the investment opportunities of each player and compute their optimal portfolio allocation. This exercise will be useful to highlight the natural trade-off between diversification and integration. Assuming agents both uncertainty and risk-averse, it would not necessarily be profitable to invest in partners which allocate great shares of capital across multiple peers (holding a more diversified portfolio).

Following this, we characterise the optimal and the stable network structures given positive costs per-link, ambiguity and risk aversion parameters ($c, \theta$ and $\lambda$ respectively). The optimal network is the structure which maximises an utilitarian welfare function for a given parametric setting, while a stable network is a structure which arises as equilibrium when each player unilaterally and optimally decides to activate new or sever existing links with other peers. The main results are as follows: for particularly high and low costs per-link the unique optimal and stable structures can only be the empty and complete graphs respectively, while for mid-range costs both the optimal and stable architecture will similarly depend on the trade-off between $\theta$ and $\lambda$ parameters. When $\theta$ is particularly high with respect to $\lambda$ we expect the formation of disconnected multiple cliques (separate fully connected clusters of nodes), while a relatively sparse connected networks when $\theta$ is low. The intuition is that when uncertainty
is relatively low (or equivalently agents are almost ambiguity neutral), the priority of risk averse agents would be to hold the most feasible diversified portfolio. This requires holding assets of multiple counterparties among those who are most diversified. When uncertainty is relatively high, ambiguity (and risk) averse agents also become concerned about the uncertainty created by indirect connections. Consequently, the most diversified firms now might be relatively less attractive due to their higher exposure to assets with uncertain returns.

Finally, we show that despite sharing similar architectures, stable and optimal networks do not necessarily coincide and could differ in link density, as each player fails to internalise the impact of her link-choice on the peers which are investing in her portfolio.²

The literature on financial network, contagion, and systemic risk is vast and we refer to Allen & Babus [2009] for an exhaustive review.³ This paper aims to contribute to the related literature on informational contagion. This literature studies the process by which information about one market may have impact on other markets.⁴ We share the general definition of complexity proposed by Caballero & Simsek [2013]. They model financial decisions of banks connected in a cross-holding network structure. In the absence of any shock (default of one or more firms), banks care only about the quality of their counterparties. However, when an unexpected shock hits the system, institutions not directly involved become concerned about the risk of being indirectly affected in a default-cascade. In other words, banks become concerned about the exposure of their counterparties to shocks on third (distant) firms. Agents have only local-knowledge, or they may not know where the shock hit and how it may propagate through the complex network of cross-exposures. The authors show that this uncertainty may trigger liquidity-precautionary behaviours yielding extended fire-sales of assets. In our setting, we also assume local-knowledge (banks observe only their direct counterparties) but we model the uncertainty about characteristics of distant firms specifically as

²Interestingly, Kovářík & van der Leij [2014] obtain similar results in a different model setting with no ambiguity. They suggest that the formation of clusters may solve at least partially the “first-order” uncertainty (risk) faced by a decision maker. In our setting, formation of clusters would actually increase risk by means of lower diversification.

³An incomplete list of papers could include, among many others, Rochet & Tirole [1996], Kiyotaki & Moore [1997], Allen & Gale [2000], Lagunoff & Schreft [2001], Freixas et al. [2000], Eisenberg & Noe [2001], Dasgupta [2004], Leitner [2005], Cifuentes et al. [2005], Lorenz et al. [2009], Allen et al. [2010], Zawadowski [2011], Demange [2012], Billo et al. [2012], Diebold & Yilmaz [2014], Dette et al. [2011], Cohen-Cole et al. [2014], Courielroux et al. [2016].

⁴See for instance the contributions Kodres & Pritsker [2002], Pavlova & Rigobon [2008], Caballero & Simsek [2013], and Alvarez & Barlevy [2015].

⁵The Executive Director of the Bank of England, Andrew G. Haldane, powerfully described the information problem faced by financial institutions at the peak of the crisis: “Knowing your ultimate counterparty’s risk then becomes like solving a high-dimension Sudoku puzzle. Links in the chain, like cells in the puzzle, are unknown—a and determining your true risk position is thereby problematic.” [Haldane et al. [2009b]].
a Knightian-type uncertainty. This allows us to work with a precise behavioural setting, such as the KMM model \cite{Klibanoff2005,Maccheroni2013}, and to study the effects of changes on risk and ambiguity aversion parameters on the networks arising in equilibrium.

In Battiston et al.\cite{Battiston2012a}, the authors study the robustness of a financial network structure and analyse two types of externalities arising from this setting. The first type is related to the immediate effect of a default of a borrower on the balance-sheet health of her lenders. The second type refers to the precautionary behaviour of firms who, after an initial default of one or more counterparties, may expect future defaults by other counterparties, given imperfect knowledge about their general exposures to risk. The banks’ decision to precautionary *run* on a counterparty is heuristic; a firm runs on an agent when she has low robustness (as measured by her *equity ratio*), and the number of defaults among her counterparties exceeds a certain threshold. The goal of our paper is different but we share the general idea that uncertainty over the exposure of counterparties may generate precautionary behaviours after an unexpected financial distress.

We believe that this paper contributes to the Economic and Finance literatures in multiple ways. First, to the best of our knowledge, this is the first paper which characterizes the risk and uncertainty faced by agents in a network as function of the structural features of the same system. Second, the model proposed relates network changes to the risk and ambiguity attitudes of the agents composing it. This may naturally open new testable hypothesis and offer new insights to policy makers alike.

We proceed as follows. In Section 2, we define the network setup. In Section 3 we describe the model, discuss the portfolio optimisation problem faced by each player for a given network structure, and characterise the optimal and stable network structures. Section 4 concludes. All the proofs are in Appendix B.

## 2 Network setup

Define with $G(N, L)$ a network of $N = \{1, \ldots, n\}$ nodes with $n \geq 3$, connected by a finite set of directed linkages $L$. For each pair of connected players $(i, j) \in N$, we indicate by $ij \in L$ a directed edge from $i$ to $j$. A link $ij \in L$ is *bilateral* if and only if $ji \in L$. The graph $G(N, L)$ is directed and is described by an adjacency matrix $A$, which is square, not necessarily symmetric, with 0 elements in the diagonal, and 1 in the $ij$-th location whenever

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6With abuse of notation, we are going to use the the terms *graph*, *digraph* and *network*, *nodes* and *players*, *links* and *edges*, interchangeably.
and such that every link in \( L \) is a sequence of distinct connected nodes. A (directed) path in \( G \) is a sequence of distinct connected nodes \( P_{i,j} = \{i_k, i_{k+1}, \ldots, i_m\} \) of order \( |P_{i,j}| \geq 2 \), where \( i_k = i, i_m = j, i \neq j, \) and \( i_k i_{k+1} \in L \) for every \( k = 1, \ldots, m - 1 \). A walk \( W_{i,j} \) is a sequence of connected nodes not necessarily distinct, or alternatively said, which can contain loops or cycles. The girth is the minimal length of a cycle. A graph is connected if there exists at least one walk \( W_{i,j} \) between any pair of nodes \((i,j) \in N\). If \( P_{i,j} \neq \emptyset \), then we call descendant nodes of \( i \) any \( k \in P_{i,j} \setminus \{i\} \) and \( i \) the father node of \( k \). A direct descendant node of \( i \) is a descendant node \( j : ij \in L \) and a direct father node of \( i \) is a father node \( j : ji \in L \). The sets of descendant and father nodes of \( i \) are respectively \( S_i(G) \) and \( F_i(G) \), while the sets of direct descendant nodes and direct father nodes of \( i \) are respectively \( S_i^d(G) \) and \( F_i^d(G) \). The out-degree of \( i \) is given by \( \delta_i^+ = |S_i^d| \), while the in-degree of \( i \) by \( \delta_i^- = |F_i^d| \). The degree of \( i \) is simply \( \delta_i = |\{S_i^d \cup F_i^d\}| \). We will make use of the definitions of local cluster presented in Fagiolo [2007]. Informally we consider a cluster any possible triangular connection between a triplet of nodes, i.e. each pair of nodes in a triplet is connected by at least one directed link. We indicate with \( c_i(G) \in [0,1] \) the clustering coefficient of node \( i \) and with \( \bar{c}(G) \) the average score in \( G \). A graph is bipartite if we can divide the set of nodes \( N \) into two independent sets, say \( N_c \) and \( N_p \) such that \( N_c \cup N_p = N \), and such that every link in \( L \) connects a node in \( N_c \) to one in \( N_p \) or vice versa. A bipartite graph is balanced if the cardinality of the two sets are the closest possible. More generally we define a set independent if there is no pair of nodes belonging to it which is connected by a link. We define \( k \)-out-regular network as a graph where \( \delta_i^+ = k > 0 \) for all \( i \in N \). The empty graph \( G^0 \) is the graph such that \( \delta_i^+ = 0 \) for all \( i \in N \), the dyad graph \( G^d \) is such that \( \delta_i^+ = 1 \) for all \( i \in N \) and \( ij \in L \iff ji \in L \), and the complete graph of \( n \) nodes, \( K_n \), is the graph where \( \delta_i^+ = n - 1 \) for all \( i \in N \). Finally, we define \( G_{n,q} \) as the (finite) set of all the directed graphs of \( n \) nodes and \( q \) number of linkages.

### 3 Model

Consider a finite set of players \( N \) of cardinality \( |N| \geq 3 \). The players are homogeneous in risk and ambiguity aversion with aversion coefficients \( \lambda \) and \( \theta \) respectively. By assuming homogeneous risk and ambiguity aversion coefficients, we aim to characterize equilibrium

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7We ruled out the possibility of self-loops, or \( ii \notin L \forall i \in N \).

8To simplify the notation, when it is clear from the context that we are referring to regularity in out-degree, we will simply report \( k \)-regular networks.

9We assume coefficients \( \lambda = -u''(w)/u'(w) \) with \( u(\cdot) \) a vN-M utility function and \( w \) a sure wealth, and \( \theta = -\phi''/\phi' \) with \( \phi \) a function which, as we will show below, captures the attitude of the DM toward ambiguity.
choices as uniquely function of the players’ location in $G(N, L)$ and not of intrinsic differences in their risk and uncertainty attitudes.

Each player is endowed with a unit capital and with a risky project (primitive asset) with normally distributed returns. Players can also hold shares of other agents’ portfolio. For any pair $(i, j) \in N^2$, $\alpha_{ij} \in [0, 1]$ indicates the share of player $i$’s capital invested on player $j$. The cross-holding structure can be represented by a (weighted) directed network $G(N, L)$, where $L$ defines the cross-holdings of the peers in $N$. Thus, $ij \in L$ describes a directed linkage from $i$ to $j$, or player $i$ owning shares of $j$ but not necessarily vice-versa.

It is assumed that for any triplet $i, j, k \in N$ such that $\{ij, jk\} \in L$ and $ik \notin L$, player $i$ faces multiple priors over the returns of player $k$’s individual project. Alternatively put, only a direct link between $i$ and $k$ could solve player $i$’s uncertainty over the expected return of $k$’s asset, i.e. the set of possible scenarios expected by $i$ when investing in player $j$’s asset which in turn is investing in $k$’s one is nonsingleton. In particular, denote by $P$ a probabilistic model (a scenario) faced by a player $i$. If $\{ij, jk\} \in L$ and $ik \notin L$, then $P$ is not unique and $i$ has prior $\mu$ on the nonsingleton bounded space $\Delta_i$ of possible models $P$. Each model defines the expected return and relative variance of agent $j$’s portfolio at least partially exposed to $k$’s assets return. If instead there is no such $k$ (either $ik \in L$ for all $k$ such that $jk \in L$ or $jk \notin L$ for all $k \neq i$), then $P$ would be unique or $\Delta_i$ singleton. This is the crucial feature of the model; in particular, it aims to model the incompleteness of the information owned by each player as function of his indirect exposures.[10]

In order to use a mean-variance model (and thus the relative extension for the ambiguity case) we assume constant absolute risk aversion utility functions, and risky primitive assets normally distributed with mean $r > 0$ and variance $\sigma^2$, and prior $\mu$ normally distributed with mean $\bar{\mu}$ and variance $\sigma^2_\mu$. In order to disentangle the impact of the link-structure on the individual choices from any other source of heterogeneity, we assume the asset variance and mean, $\sigma^2$ and $r$ respectively, as constant across the players.

Finally, each player $i \in N$ observes the sub-graph $G_i \subseteq G$ composed by players $j \in G_i$ distant at most two links ($d_{ij} \leq 2$). In other words, we assume semi-myopic players. This restriction is common in network theory (see Galeotti et al. [2010] and Gallo [2012]), and empirically supported (see Fafchamps et al. [2010]).[11]

Remark 1. The players in $G(N, L)$ are “naive”. Alternatively said, for each triplet of players $i, j, k \in$.

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[10] As previously stressed, this is qualitatively in line with the definition of complexity given by Caballero & Simsek [2013].

[11] Incomplete information about an agent two links away from a player could be also interpreted as the player’s inability to observe the whole cross-holding network.
such that \( \{ij, jk\} \in L \) and \( ik \notin L \), \( i \) does not “refine” her prior \( \mu \) taking the exposure choice of \( j \) toward \( k \) as informative signal.

This remark highlights an important element of the model and under certain degrees could be interpreted as an extension of the players’ bounded rationality which we are implying. We point it out that assuming risk and ambiguity attitude of each agent as private information might be a sufficient condition to observe naive players; under incomplete information about risk and ambiguity preferences of the rest of the peers, agents are not able to retrieve certain information about the characteristics of portfolios of indirect partners from the choices of their direct partners. Future research could investigate this element and enrich the model.

### 3.1 The ambiguity model setting

We start by fixing \( G(N, L) \) and allowing each player \( i \in N \) to optimally choose \( \alpha_{ij} \in [0, 1] \) for all \( j \) such that \( ij \in L \). This would consist of solving a one-period constrained optimization problem.\(^{12}\) To formally define the problem, we make use of a similar setting to that proposed by Maccheroni et al.\(^{13}\) [2013]. In particular, consider the probability space \((\Omega, \mathcal{F}, P)\) and let \( L^2 = L^2(\Omega, \mathcal{F}, P) \) be the Hilbert space of square integrable random variables on \( \Omega \). The gross-return on an asset \( z \) is \( r_z \in L^2 \), the vector of returns is \( \mathbf{r} \), and the vector of portfolio weights of player \( i \) is \( \alpha_i \), composed by \( n \) elements \( \alpha_{ij} \) for each \( j \) such that \( d_{ij} \leq 1 \).\(^{13}\) The end-of-period wealth of agent \( i \) holding a portfolio \( \alpha_i \), denoted by \( r_{\alpha_i} \), is computed as

\[
r_{\alpha_i} = \alpha_{ii} r_i + \sum_{j \in S_i^d} \alpha_{ij} r_{\alpha_j}
\]

with \( \alpha_{ii} \) the shares invested by \( i \) in her individual project with expected return \( r_i \), and \( \alpha_{ij} \) the share invested in player \( j \neq i \) with expected return conditional to \( j \)’s portfolio return \( r_{\alpha_j} \). At all times it must be true that \( \alpha_{ii} + \sum_{j \in S_i^d} \alpha_{ij} \leq 1 \) with shares \( \alpha_{ij} \in [0, 1] \) for any \( j \).\(^{14}\)

Given the space \( \Delta \) of possible models \( P \), \( E[r] : \Delta \to \mathbb{R} \) is the random variable that associates the expected value \( E_P[r] \) to each possible \( P \). We use the KMM decision model setting proposed by Klibanoff et al.\(^{15}\) [2005], where a player \( i \) ranks any portfolio \( \alpha_i \) as following. For a non-singleton interval \( I \subseteq \mathbb{R} \) of monetary outcomes, \( i \) ranks \( \alpha_i \) according to the functional

\[\text{rank}(\alpha_i) = E_P[r_i | \alpha_i] \]
form $V : L^\infty(I) \to \mathbb{R}$,

$$V(\alpha_i) = \int_\Delta \phi \left( \int_\Omega u(\alpha_i) d\pi \right) d\mu, \quad \forall \alpha_i \in L^\infty(I),$$

with $\mu$ a prior over $\Delta$ with bounded support, $\pi$ is a probability measure over the state space $\Omega$, $u : I \to \mathbb{R}$ a standard vN-M utility function and $\phi : u(I) \to \mathbb{R}$ a continuous and strictly increasing function. Finally, we use the Arrow-Pratt approximation for the KMM setting proposed by Maccheroni et al. [2013]. The authors parsimoniously show that, under the KMM model setting, the extension of the mean-variance model under ambiguity is summarised by

$$U(\alpha_i; \lambda, \theta) = E_P(\alpha_i) - \frac{\lambda}{2} \sigma^2_P(\alpha_i) - \frac{\theta}{2} \sigma^2_P(E(\alpha_i))$$

with $\sigma^2_P(\alpha_i)$ and $\sigma^2_P(E(\alpha_i))$ respectively the variances measuring the risk and uncertainty as perceived by the DM evaluating $\alpha_i$.

### 3.2 Diversification and Integration

We are going to clarify what we mean by diversification, integration, and ambiguity, with reference to a network structure. Consider the $n \times n$ square matrix $W$ composed by $ij$-elements $\alpha_{ij}$ which describes the cross-holding structure $G$. Define the matrix $D = W + W^2$ and call $\mathbf{1}$ the vector of all 1s. The matrix $D$ captures both the direct and indirect exposures between agents. We can compare two structures and two nodes in terms of diversification and integration as follows:

**Definition 1.** A structure $G$ is more diversified than $G'$ if and only if

- $D_{ij} \leq D'_{ij}$ for all $i$ and $j$ such that $D_{ij}$ and $D'_{ij}$ are strictly positive with strict inequality for some ordered pair $(i, j)$, and

- $D_{ij} > D'_{ij} = 0$ for some $i$ and $j$.

A structure $G$ is more integrated than $G'$ if and only if

$$W \cdot \mathbf{1} - \alpha_i > W' \cdot \mathbf{1} - \alpha_i$$

with $\alpha_i$ the vector of $\alpha_{ii}$ elements.
In other words, for a fixed level of integration, the level of diversification of $G$ is relatively higher than the one of $G'$ if the total share invested in other organizations is spread among relatively more distinct institutions. On the other hand, for a fixed level of diversification, the level of integration is relatively higher if the total share invested in other agents is higher.

Observe that using this particular definition of diversification, we can "discount" for the presence of clusters in the network; when a node $i$ is part of a cluster of three nodes such as $\{ij, jk, ik\}$, he is investing both directly and indirectly in $k$’s portfolio, thus he may get lower diversification than in the case $\{ij, jq, ik\}$. In particular, this is an important difference between our definition and the one proposed by Elliott et al. [2014] which instead does not take into account indirect linkages. The example below should clarify this point.

Define $1_{ij}$ the indicator function such that $1_{ij} = 1$ if $W_{ij} = 0$ and $1_{ij} = 0$ if $W_{ij} > 0$.

**Definition 2.** A structure $G$ with matrix of cross-holdings $W$ is non-ambiguous if and only if for any pair of distinct nodes $(i, j)$

$$W_{ij} = 0 \Rightarrow W_{ij}^2 = 0$$

otherwise it is ambiguous. Moreover, $W$ is more ambiguous than $W'$ if and only if for any pair of distinct nodes $(i, j)$

$$\sum_{(i,j) \in N^2} 1_{ij} W_{ij}^2 \geq \sum_{(i,j) \in N^2} 1_{ij} W'_{ij}^2$$

with strict inequality for at least one pair.

In other words, a structure is ambiguous if there are nodes which are indirectly exposed to agents with uncertain returns; if in a given network structure there is no pair of nodes (geodesically) distant more than one link, then no player is facing uncertain scenarios and thus the whole structure is defined as non-ambiguous. Note that in a similar way we can compute the total exposure to uncertain scenarios of a single player $i \in N$ as $\sum_{j \in N} 1_{ij} W_{ij}^2$.

Observe that, according to these definitions, a complete network is non-ambiguous and the most diversified structure possible; a complete structure will exhaust any possible investment possibility and therefore it will be maximally diversified, without exposing nodes to uncertain scenarios.

\[^{15}\text{Similar definition can be applied at node level.}\]
3.2.1 Example

Consider the graphs $G$ and $G'$ in Figure 1. The networks have same number of nodes but different link-structure as defined by the cross-holding matrices $W$ and $W'$ respectively,

\[
W = \begin{pmatrix}
\alpha_{11} & \alpha_{12} & \alpha_{13} & 0 \\
0 & \alpha_{22} & 0 & \alpha_{24} \\
0 & 0 & \alpha_{33} & 0 \\
0 & 0 & 0 & \alpha_{44}
\end{pmatrix}
\quad W' = \begin{pmatrix}
\alpha_{11} & \alpha_{12} & \alpha_{13} & 0 \\
0 & \alpha_{22} & \alpha_{23} & 0 \\
0 & 0 & \alpha_{33} & 0 \\
0 & 0 & 0 & \alpha_{44}
\end{pmatrix}
\]

Assume constant integration when positive. According to the definitions proposed by Elliott et al. [2014], the two structures would be equally diversified, while in our setting $G$ is more diversified than $G'$.

![Figure 1](image_url)

**Figure 1:** The two structures differs only in the link from player 2. $G'$ is non-ambiguous but less diversified than $G$, which is ambiguous due to the exposure of player 2 to player 4.

Their respective $D$ and $D'$ matrices (recall that $D = W + W^2$) are

\[
D = \begin{pmatrix}
\alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{12}\alpha_{24} \\
0 & \alpha_{22} & 0 & \alpha_{24} \\
0 & 0 & \alpha_{33} & 0 \\
0 & 0 & 0 & \alpha_{44}
\end{pmatrix}
\quad D' = \begin{pmatrix}
\alpha_{11} & \alpha_{12} & \alpha_{13} + \alpha_{12}\alpha_{23} & 0 \\
0 & \alpha_{22} & \alpha_{23} & 0 \\
0 & 0 & \alpha_{33} & 0 \\
0 & 0 & 0 & \alpha_{44}
\end{pmatrix}
\]

In $G$ and $G'$ respective players own the same number of links. However, in $G'$ player 2 forms a cluster by connecting to player 3, which is already a counterpart of player 1. The graph $G$ is also more ambiguous than $G'$ since

\[
\sum_{(i,j)\in N^2} 1_{ij}W_{ij}^2 = \alpha_{12}\alpha_{24} > 0 = \sum_{(i,j)\in N^2} 1_{ij}W'_{ij}^2
\]
or in \( G \), player 1 is exposed to uncertainty via player 2 which is connected to player 4.

### 3.3 Portfolio optimization problem

We are going to describe in more detail the capital allocation problem solved by an agent for a fixed link-structure. To illustrate the impact of diversification and exposure to uncertain scenarios, we discuss in light of our assumptions one of the comparative static exercises originally proposed by Maccheroni et al. [2013]. In particular, we study the optimal portfolio allocation between a purely risky asset and an ambiguous one. Consider a triplet of nodes \( i, j, k \), where \( i \) is uniquely connected to \( j \) and \( j \) uniquely to \( k \) (Figure 2). In this context, for player \( i \), the purely risky asset is represented by his own private project while agent \( j \)’s portfolio, which is exposed to agent \( k \)’s uncertain scenarios, represents the ambiguous alternative. The portfolio problem of player \( i \) could be summarized by the following optimization problem,

\[
\max_{\alpha_i \in \mathbb{R}^2} U_i(\alpha_i; \lambda, \theta) = \max_{\alpha_i \in \mathbb{R}^2} \left( E_P(\alpha_i) - \frac{\lambda}{2} Var_P[\alpha_i] - \frac{\theta}{2} Var_{\mu}[E[\alpha_i]] \right)
\]

subject to the constraint \( \alpha_{ii} + \sum_{j \in S_i^d} \alpha_{ij} \leq 1 \). \( Var_P[\alpha_i] = [\sigma_P(r_i, r_{\alpha_j})]_{j=1}^k \) is the variance-covariance matrix of returns under a specific model \( P \), and \( Var_{\mu}[E[\alpha_i]] = [\sigma_{\mu}(E(r_i), E(r_{\alpha_j}))]_{j=1}^k \) is the variance-covariance matrix of expected returns under the prior \( \mu \) on the space \( \Delta \) of possible scenarios. For player \( i \), the solution to the problem is given by the vector

\[
\alpha_i^* = \left( \lambda Var_P[\alpha_i] + \theta Var_{\mu}[E[\alpha_i]] \right)^{-1} \cdot E_P[r_{\alpha_j}]
\]

where, \( E_P[r_{\alpha_j}] = (E_P[r_i], E_P[r_{\alpha_j}]) \) is the vector of expected returns under \( P \). The optimality condition requires

\[
\lambda \begin{bmatrix} \sigma_P^2(r_i) & \sigma_P(r_i, r_{\alpha_j}) \\ \sigma_P(r_i, r_{\alpha_j}) & \sigma_{\mu}^2(r_{\alpha_j}) \end{bmatrix} \begin{bmatrix} \alpha_{ii} \\ \alpha_{ij} \end{bmatrix} + \theta \begin{bmatrix} 0 & 0 \\ 0 & \sigma_{\mu}^2(E(r_{\alpha_j})) \end{bmatrix} \begin{bmatrix} \alpha_{ii} \\ \alpha_{ij} \end{bmatrix} = \begin{bmatrix} E[r_i] \\ E[r_{\alpha_j}] \end{bmatrix}
\]

In order to disentangle the impact of assets’ returns and their variance, we assume \( E[r_i] = E[r_{\alpha_j}] = 1 \); any variation in the capital allocation will be the result of the trade-off between the two type of variances or by taste preferences over risk and uncertainty. Each individual project has homogeneous variance \( \sigma_P^2(r_i) > 0 \) and covariance \( \sigma_P(r_i, r_{\alpha_j}) = 0 \) with \( r_i \) and \( r_{\alpha_j} \) indicating respectively \( i \)’s project and \( j \)’s portfolio rates of return. Define \( \psi \equiv (\alpha_{ii}/\alpha_{ij}) \). Rearranging, we obtain
\[ \psi = \frac{1}{\sigma_P^2(r_i)} \left( \sigma_P^2(r_{\alpha_j}) + \frac{\theta}{\lambda} (\sigma_{\mu}(E[r_{\alpha_j}])) \right) \]

We can compute the marginal changes with respect to the two variances and the risk and ambiguity-aversion coefficients,

\[ \frac{\partial \psi}{\partial \sigma^2_\mu} = \frac{\theta}{\lambda} \frac{1}{\sigma_P^2(r_i)} > 0 \quad \frac{\partial \psi}{\partial \theta} = \frac{1}{\lambda \sigma_P^2(r_i)} > 0 \]

\[ \frac{\partial \psi}{\partial \sigma^2_P(r_{\alpha_j})} = \frac{1}{\sigma_P^2(r_i)} > 0 \quad \frac{\partial \psi}{\partial \lambda} = -\frac{\theta}{\lambda^2 \sigma_P^2(r_i)} < 0 \]

Observe that the change in ambiguity aversion and variance over the set \( \mu \) share the same positive sign (\( sgn(\frac{\partial \psi}{\partial \sigma^2_\mu}) = sgn(\frac{\partial \psi}{\partial \theta}) \)). Therefore, we can only study variations due to \( \theta \) to assess the impact of a variation on the support of \( \mu \).

For any generic \( G(N, L) \), define by \( K_j \) the subset of nodes which are direct descendant of \( j \) but not of \( i \). Formally, \( K_j \subseteq S_j(G) \) such that \( K_j \cap S_i(G) = \emptyset \). Suppose \( ij \in L \). The total indirect exposure of \( i \) to any uncertain scenario via agent \( j \) is \( \tau_j \equiv \sum_{q \in K_j} \alpha_{ij} \). The uncertainty as perceived by \( i \) is measured by \( \sigma^2_{\mu}(\cdot) \) which monotonically increases with the exposure of \( j \) to \( q \in K_j \) players (\( \partial \sigma^2_\mu/\partial \tau_j > 0 \)). Keeping constant \( j \)’s integration level, it is intuitive to see that if \( j \) spreads her total exposure across more players (increasing her portfolio’s diversification), she indirectly increases \( i \)’s portfolio’s diversification level as well (\( \partial \sigma^2_P(r_{\alpha_j})/\partial \delta_j^+ < 0 \)).

Therefore, the marginal change of cross-exposure of \( i \) due to either a change in risk-aversion or a change of the size of \( \delta_j^+ \) shares the same negative sign for a fixed level of \( j \)’s integration (\( sgn(\frac{\partial \psi}{\partial \delta_j^+}) = sgn(\frac{\partial \psi}{\partial \lambda}) \)). We then obtain the following predictions. Given a directed graph \( G(N, L) \) and a triplet of players \( i, j, k \in N \) such that \( \{ij, ik\} \in L \),

- If \( \tau_j \geq \tau_k \), then \( \alpha_{ij}^* \leq \alpha_{ik}^* \) for high enough \( \theta \) and constant diversification level.
- If \( \delta_j^+ \geq \delta_k^+ \) and \( \tau_j \approx \tau_k \), then \( \alpha_{ij}^* \geq \alpha_{ik}^* \), for positive \( \lambda \) and small enough \( \theta \).

All things being equal and holding \( j \)’s diversification constant, if \( j \)’s exposure over ambiguous assets rises, player \( i \) optimally decreases her cross-exposure to \( j \) for higher ambiguity aversion levels. On the other hand, holding \( j \)’s and \( k \)’s integration levels constant, if \( j \) spreads

\[^{16} \text{In Appendix we show that, all things being equal, the out-degree of a node positively affects his diversification level, therefore if } j \text{'s diversification increases with } \delta_j^+, \text{ the portfolio variance of node } i \text{ decreases too.} \]
her exposure across more agents than \( k \), player \( i \) optimally would increase her exposure toward \( j \) relative to \( k \). In Figure 2, player 2 diversifies her investment owning shares of player 3’s assets and thus, player 2’s assets may become particularly profitable to player 1 for relatively low \( \theta \) and strictly positive \( \lambda \). However, by increasing \( \theta \), \( i \)’s exposure to 2 decreases as a result of the increased sensitivity to the uncertainty over the returns of player 3’s assets.

![Figure 2](image)

**Figure 2**: Player 1’s optimal exposure on player 2 as function of \( \theta \) and \( \lambda \), and player 2’s optimal exposure on player 3.

### 3.4 Optimal network structure

Having discussed the capital allocation problem faced by a player in a given network, we discuss the structure which maximizes the aggregate welfare. Consider the problem faced by a central planner aiming to maximise the following utilitarian welfare function

\[
V(G, \lambda, \theta, c) = \sum_{i \in N} U_i(G, \lambda, \theta) - c \sum_{i \in N} \delta_i^x
\]

with \( c \geq 0 \) constant marginal cost per-link which is paid for each ordered pair \((i, j)\) such that \( \alpha_{ij} > 0 \). The central planner chooses the portfolio of each player \( i \in N \) subject to the unit capital constraint per player. In other words, he chooses an adjacency matrix where \( \sum_{j \neq i} \alpha_{ij} \leq 1 \) for each row \( i \). A structure \( G^*(N, L) \) is optimal if and only if \( V(G^*, \lambda, \theta, c) \geq V(G', \lambda, \theta, c) \) for all \( G' \neq G^* \).

Assume positive \( \lambda \) and consider two nodes \( j \) and \( q \) equally diversified. Define \( G' = G + ij \) and \( G'' = G' + iq \), and \( \Delta f(G) = f(G') - f(G) \) for any generic function \( f \). We know that when \( \theta = 0 \), then \( \Delta U_i(G, \lambda, \theta) > 0 \) and \( \Delta U_i(G', \lambda, \theta) < \Delta U_i(G, \lambda, \theta) \) for all nodes \( i \) activating new links \( ij \) and \( iq \), and similarly for any node \( z \) investing in \( i \)’s portfolio. In words, when players
are ambiguity-neutral, the activation of a new link by an agent \( i \) increases diversification of the portfolios of \( i \) and of his parent nodes \( z \in F_i \). Thus, the complete network, \( K_n \), always guarantees the highest diversification for any \( \lambda > 0 \) and \( \theta \geq 0 \). This also means that there always exists a minimal cost level \( c \geq 0 \) such that the only optimal network is the complete one, \( G^* = K_n \). Moreover, due to decreasing returns to scale, a marginal link increases relatively less the portfolios’ diversification when \( i \) already invests in many agents. Therefore, we can always assume trivial maximal cost \( \bar{c} \) such that for any \( c \geq \bar{c} \), the benefits from diversification are smaller than the cost to sustain any link, i.e. the empty network is the only optimal structure. Finally, define the intermediary cost \( c' \in (\underline{c}, \bar{c}) \) as the cost such that \( \Delta V(G, \lambda, \theta, c') = c' \) with \( G \) such that \( \overline{cl}(G) = 0 \) and \( \overline{cl}(G') > 0 \) for any \( ij \notin L \). In other words, \( c' \) is equal to the marginal benefit yield by a link of a complete bipartite graph \( G \). Recall the ratio \( \varphi \equiv \lambda/\theta \).

**Proposition 1.** Consider the utilitarian welfare function \( V(G, \lambda, \theta, c) \). There exists a level \( \varphi^U \) such that for any \( \varphi \geq \varphi^U \) and cost per link \( c \) high enough, the optimal structure \( G^* \) is a q-partite digraph.

Moreover, there exists a level \( \varphi^L < \varphi^U \) such that for any \( \varphi \leq \varphi^L \), \( G^* \) is composed by multiple disconnected cliques of size monotonically decreasing with \( c \).

The result states that a hypothetical central planner maximising the utilitarian welfare function \( V(G, \lambda, \theta, c) \) would choose for relatively high ratios \( \lambda/\theta \) a sparse network structure, while a structure composed by multiple cliques for relatively small \( \lambda/\theta \) ratios. In Figure 3 each point of the plane represents the marginal change in uncertainty and risk, labelled \( B \) and \( |A| \) respectively, due to the activation of a new link (red link on the graphs). The points on the line with slope \( \bar{\lambda}/\bar{\theta} \) define links with zero marginal change (\( B \) and \( |A| \) offset each other). A structure is optimal if and only if any new link-activation is described by points in the shaded area, i.e. negative marginal change. Decreasing the marginal cost per link shrinks the shaded area allowing the activation of links with relatively low diversification impact and high exposure to uncertainty.

We remark that by assuming a generic welfare function such as \( V(\lambda, \theta, c) \), we are able to describe distinct central planner’s objectives by simply assuming different parametric settings \((\lambda, \theta)\). Define \((\lambda^c, \theta^c)\) the parametric setting considered by the central planner. The parameters \( \lambda^c \) and \( \theta^c \) should not necessarily coincide with \( \lambda \) and \( \theta \), the risk and uncertainty attitudes of the agents in \( N \). A setting such as \( \lambda^c > 0 \) and \( \theta^c \to 0 \) for example would describe a central planner uniquely interested to construct the most diversified structure for a given marginal cost per-link. As we will discuss later, differences between \((\lambda^c, \theta^c)\) and \((\lambda, \theta)\) of
individual agents might amplify possible structural differences between optimal and stable networks. If not stated otherwise, it is assumed hereafter that \((\lambda^c, \theta^c) = (\lambda, \theta)\).

### 3.4.1 Example

Consider again the cross-holding network structures of the previous example where \(G\) was more diversified but more ambiguous than \(G'\). In terms of social welfare, we can show that \(V(G, \lambda, \theta, c) \geq V(G', \lambda, \theta)\) only for particularly high ratios \(\lambda/\theta\). Observe that, since \(U_i(G, \lambda, \theta) = U_i(G', \lambda, \theta)\) for \(i = \{3, 4\}\), and \(\sum_i \delta_i^+\) is the same for the two structures, we have \(V(G, \lambda, \theta, c) \geq V(G', \lambda, \theta, c)\) if and only if

\[
U_1(G, \lambda, \theta) - U_1(G', \lambda, \theta) + U_2(G, \lambda, \theta) - U_2(G', \lambda, \theta) \geq 0
\]

and since the expected returns are the same, this implies

\[
\lambda[\Delta \sigma^2_P[r_{\alpha_1}] + \Delta \sigma^2_P[r_{\alpha_2}]] + \theta[\Delta \sigma^2_\mu[E[r_{\alpha_1}]] + \Delta \sigma^2_\mu[E[r_{\alpha_2}]]] \geq 0
\]

Observe that \(\Delta \sigma^2_P[r_{\alpha_2}] = 0\) and \(\Delta \sigma^2_\mu[E[r_{\alpha_2}]] = 0\) since agent 2 faces the same variances in the two structures. Moreover, \(\Delta \sigma^2_\mu[E[r_{\alpha_1}]] = -\sigma^2_\mu[E[r_{\alpha_1}]]\) since in \(G'\) agent 1 does not face
uncertain scenarios. Thus, we can simplify the expression as

$$\lambda[\Delta \sigma^2_P[r_{a_1}]] - \theta[\sigma^2_\mu[E[r_{a_1}]]] \geq 0$$

and therefore $V(G, \lambda, \theta, c) \geq V(G', \lambda, \theta, c)$ if and only if

$$\frac{\lambda}{\theta} \geq \frac{\sigma^2_\mu[E[r_{a_1}]]}{\Delta \sigma^2_P[r_{a_1}]}$$

where $\Delta \sigma^2_P[r_{a_1}] > 0$. In words, for a given $\lambda > 0$, network $G'$ leads to higher welfare only for positive and high enough ambiguity aversion parameters $\theta$ (see Figure 4). Intuitively, since player 1’s diversification is lower in $G'$ than in $G$, she would prefer it only if particularly averse to uncertainty.

![Figure 4: The structure $G'$ is non-ambiguous and thus its welfare level, $V(G', \lambda, \theta, c)$, is not affected by changes in $\theta$. For $\lambda = 2$ and from relative small positive $\theta$ values ($\theta \approx 0.45$), $V(G', \lambda, \theta, c)$ is higher than $V(G, \lambda, \theta, c)$, the welfare related to the structure $G$.](image)

### 3.5 Stable network structures

We are going to analyse the network structures which could arise when we allow players to optimally rewire their linkages. Assume that the activation(cut) of a new(old) directed linkage is a unilateral choice of the player activating(severing) the link and the marginal cost $c$ is paid entirely by the proposer of the link.\(^{17}\) We describe the game $\Gamma = \langle N, \succeq, S \rangle$ as follows. Each player $i$ in $N = \{1, \ldots, n\}$ simultaneously chooses an allocation $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{ii-1}, \alpha_{ii+1}, \ldots, \alpha_{in})$.

\(^{17}\)To justify this setting, recall that receiving one linkage from a player would not affect the relative expected utility of the receiver, thus the unilateral choice assumption seems reasonable. Recall that this would not exclude the fact that a new linkage could indirectly affect another player’s expected utility.
which is a vector describing her portfolio composition with element $\alpha_{ij} \in [0, 1]$ such that $\sum_{j \neq i} \alpha_{ij} \leq 1$. We have $\alpha_i \in S_i = \{0, 1\}^{n-1}$ with $S_i$ the set of strategies of player $i$ and thus $S = \times_{i \in N} S_i$ the set of strategies of all the players. The payoff of player $i$ is given by [1]. We define a stable equilibrium network structure as follows:

**Definition 3.** A directed graph $G(N, L)$ is stable with respect to $U(G; \lambda, \theta)$ if as a result of the link-rewiring process

1. $U_i^*(G; \lambda, \theta) - c\delta_i^+ \geq U_i^*(G - ij; \lambda, \theta) - c(\delta_i^+ - 1)$ for all $ij \in L$, and
2. $U_i^*(G; \lambda, \theta) - c\delta_i^+ > U_i^*(G + ij; \lambda, \theta) - c(\delta_i^+ + 1)$ for all $ij /\in L$.

with $U_i^*$ the indirect utility of player $i$. In other words, a structure $G(N, L)$ is stable if each node does not find it profitable to sever any of his active links to other nodes and to activate a new one.

Observe that, given our assumptions, the activation of any new link does not affect the expected return of the activating player’s portfolio. On the other hand, any link modification could impact his portfolio’s variance and the variance over the possible scenarios. Recall $G' = G + ij$. Then, it is profitable to activate a new link if and only if $\Delta U_i^*(G; \lambda, \theta) \geq c$, or $\lambda \Delta \sigma_\mu^2(r_{ai}) + \theta \Delta \sigma_\mu^2(E(r_{ai})) \geq c$, where $\Delta \sigma_\mu^2(r_{ai}) > 0$ and $\Delta \sigma_\mu^2(E(r_{ai}))$ has an ambiguous sign since it depends on the link-receiver node’s characteristics, i.e. whether it is exposed to uncertain scenarios or not. Define $c^u$ as the cost such that $c^u = \Delta U_i(G^0; \lambda, \theta)$ where $G^0$ is the empty network. Again, given the domain of $c$, we can always get trivial cost levels $c \geq c^u$ such that it is not beneficial for a node to activate any link, i.e. the unique stable network is the empty graph. Denote by $c^l \geq 0$ a lower bound for $c$ such that for any cost lower than $c^l$, the complete network is a stable equilibrium [18]. Similarly to the previous analysis, define the intermediary cost $c' \in (c^l, c^u]$ such that $\Delta U_i(G, \lambda, \theta) = c'$ with $G$ a complete bipartite graph. We can present the following result:

**Proposition 2.** Assume constant integration level. A stable network $G^*$ exists and for levels $\phi \geq \phi^U$ and cost $c$ high enough, $G^*$ is a $q$-partite digraph.

Moreover, there exists a level $\phi^L$ such that for any $\phi \leq \phi^L$, $G^*$ is composed by multiple cliques of size monotonically decreasing with $c$.

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[18] It is straightforward to see that this lower bound always exists; if we set $c_l = 0$ the complete network is always a stable equilibrium.
Despite the similarity between the architectures of stable and optimal network structures, the following Corollary states the restrictive conditions under which they would coincide.

**Corollary 1.** Suppose \( \lambda > 0 \) and \( \theta \geq 0 \). The stable network structure is optimal, \( G^s = G^* \), only if the following condition holds for any \( i,j \notin L \):

\[
-\lambda \sum_{q \in F_i} \Delta \sigma^2_p [r_{a_q}] - \theta \sum_{q \in F_i} \Delta \sigma^2_\mu [E[r_{a_q}]] = 0
\]  

(4)

On the other hand, if the left-hand side of (4) is positive, then \( G^s \) is less dense than \( G^* \), while more dense when the left-hand side of (4) is negative.

The results state that for intermediate costs there always exists a stable network structure, \( G^s \), which is most likely different in density than \( G^* \) but not in architecture; the two structures coincide only when the negative and positive externalities of a link activation balance out.

This comes from the fact that each player fails to internalize the impact of her link-choice on other peers (parent nodes). For instance, for \( c \) small enough \((c \in (c^l, c^u))\) and \( \theta \) small enough, the (4) is positive and thus we expect \( G^* \) to be more dense than \( G^s \). This is a consequence of the fact that when \( \theta \) is relatively small, uncertainty impacts less the expected utility of the players, thus diversification has more weight. When individual players choose to pool risk by investing in other institutions, they do not take into account the positive externalities of their choice, in terms of greater diversification, on players investing in their portfolio. This would yield to under-investment relative to the optimal level.

Despite the density, the two structures share the general architecture; if the marginal cost per link is smaller than the level \( c^l \), the complete graph will be both stable and optimal. Since the marginal utility decreases with respect to the out-degree, the level \( c^l \) decreases with \( n \), i.e. if a complete network of \( n \) nodes is stable at cost \( c^l \), then it would be also stable for smaller \( n \) but not necessarily vice versa. If risk averse players are not particularly ambiguity averse and cost per-link is \( c \in (c^l, c^u) \), then any stable network with less than \( n/4 \) links must not contains cycles strictly greater than two. In other words, players activates only the links which diversify the most (Figure 5a). Finally, if players are particularly ambiguity averse, and cost \( c \) such that \( c \in (c^l, c^u) \), sparse networks which create uncertainty cannot be neither stable nor optimal; stable and optimal networks can only be composed by isolated cliques of nodes (Figure 5b).

\[\text{19}\] Trivially, this is always the case if the stable graph is complete.
As previously remarked, central planner’s objectives may not necessarily coincide with those of the individual agents. In such a case, the differences between optimal and stable networks might be even larger. For example, suppose \( \lambda > 0 \) and \( \theta > 0 \), while \( \lambda^c > 0 \) but \( \theta^c \to 0 \). This is equivalent to the case of a central planner uniquely interested in creating the most diversified structure for a given cost \( c \) per-link. Condition (4) reduces to

\[
-\lambda c \sum_{q \in F_i} \Delta \sigma^2_P[r_{aq}] = 0
\]

which is clearly never satisfied. In other words, stable and optimal networks would never coincide, and the optimal networks would always be more dense than the stable ones.

Finally, we remark that for intermediate values of \( \varphi \) and \( c \) we are not able to define the architecture of a stable network since this will depend on the specific combinations of the model’s parameters \( c, \theta, \) and \( \lambda \). Suppose intermediate cost \( c \in (c^L, c^U) \) and \( \varphi \in (\varphi^L, \varphi^U) \). A node \( i \in N \) will activate a link \( ij \notin L \) if and only if \( \Delta U_i(G, \lambda, \theta) = -\lambda \Delta \sigma^2_P[r_{aq}] - \theta \Delta \sigma^2_\mu[E[r_{aq}]] \geq c \). We can rewrite this condition as

\[
\varphi \geq -\frac{1}{\Delta \sigma^2_P[r_{aq}]} \left( \frac{c}{\theta} + \Delta \sigma^2_\mu[E[r_{aq}]] \right)
\]

where \(-1/\Delta \sigma^2_P[r_{aq}] > 0\). Since \( \varphi > 0 \), if \( c/\theta + \Delta \sigma^2_\mu[E[r_{aq}]] \leq 0 \), the condition (5) holds; if \( c/\theta \) is small enough and \( \Delta \sigma^2_\mu[E[r_{aq}]] < 0 \), then the new link is beneficial in terms of both

(a) Tripartite graph  
(b) Three non-connected cliques

Figure 5
greater diversification and lower exposure to uncertainty. This could happen in particular
when the new link \( ij \) is such that \( \text{cl}(G') > \text{cl}_i(G) \), or \( ij \) strictly increases the closure degree of
node \( i \). On the other hand, if \( c/\theta + \Delta \sigma^2 \mu[E[r_{\alpha_q}]] > 0 \), then we can only say that condition (5)
holds for relatively small \( c/\theta \) and \( \Delta \sigma^2 \mu[E[r_{\alpha_q}]] \), and relatively high \( \Delta \sigma^2 \mu[r_{\alpha_q}] \); to be profitable,
the marginal cost of the new link \( ij \) should be low enough and the exposure to uncertain
scenarios due to \( ij \) should not offset its positive impact due to higher diversification.

4 Conclusion

We model the formation of stable cross-holding network structures by risk and ambiguity
averse players and compare them to the optimal ones. These networks naturally define the
diversification and integration levels experienced by each agent composing them. While a
complete directed structure always represents the best architecture in terms of diversification
(maximal) and uncertainty (minimal), sparse networks are relatively diversified structures
but may expose the agents to high degrees of ambiguity. We show that sparse networks such
as \( q \)-partite ones are generally optimal and stable when agents are risk averse but not particularly
averse to ambiguity, and when cost per-link are high enough. When agents are both risk
and ambiguity averse, stable and optimal networks are composed by disconnected cliques.
Finally, stable and optimal networks generally do not coincide despite sharing similar architectures. This happens since players fail to internalize the cost and benefits of each link they
form on other peers investing on them.

The results presented could open interesting testable hypotheses about some of the styl-
ized facts observed during the recent financial crisis. For example, the benchmark model
suggests that during a time of turmoil in interbank market, uncertainty over the counterpart
investment’s quality and in particular its exposure to risky partners could affect her capital
allocation choice in a specific way; Among other things, we may expect a decrease in banks’
exposures to institutions which are more integrated with banks from non-domestic markets
(intermediary banks), an increasing exposure toward less integrated domestic banks, the for-
mation of dense smaller isolated components, and a more modest variation of exposure to-
ward partners within a cluster[20]. This could be particularly relevant for its policy implica-
tions; we believe that in order to propose effective policies, any regulation which focuses

[20] This is partially in line with some of the findings of Tabak et al. [2014] for the Brazilian interbank network
and of Minoiu & Reyes [2013] for the global banking network. Interestingly, we could also reinterpret our
results in the light of the classic work Granovetter [1973]; In our setting, weak links are those exposing a player
to uncertain scenarios while strong ties are those which do not create uncertainty.
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on the systemic impact of interconnected financial institutions should anticipate possible endogenous structural modifications due to changes in both risk and uncertainty. This could be also important to prevent potential *ex ante* moral hazard behaviours, reducing firms’ incentives to become systemic in the first place.\(^{21}\)

\(^{21}\)See for instance Acharya *et al.* [2014] and Farhi & Tirole [2009].
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A Appendix

A.1 Diversification: line Vs cluster of 3 nodes

We start showing how the variance within a scenario faced by a player \(i\) decreases with her out-degree \(\delta_i^+\). For simplicity, assume that all her direct partners are identical and isolated. Given the rest of the assumptions and assumed that the total integration is fixed at \(\tau \in (0, 1)\), we can write \(i\)’s portfolio variance as

\[
\text{Var}_P[\alpha_{ri}] = \sigma_P^2[\alpha_{i1}^2 + \sum_{j \neq i} \alpha_{ij}^2 \alpha_{jj}^2],
\]

which is minimised when \(\alpha_{ii} = \alpha_{ij} = \frac{1}{(\delta_i^+ + 1)}\) for all \(j\). Therefore, \(\sigma_P^2[\alpha_{ri}]\) decreases with \(\delta_i^+\) at decreasing rate.

![Figure 6](image)

Now we illustrate under which conditions a marginal link which creates a cluster among a triplet of nodes increases the portfolio diversification of the player activating the link. Consider three nodes, 1, 2, and 3, connected as a line \(\{12, 23\}\) (Figure 6a). Variance of player 1’s portfolio, \(\sigma_P^2[\alpha_{r1}]\), will be

\[
\sigma_P^2[\alpha_{r1}] = \sigma_P^2[\alpha_{11}^2 + \alpha_{12}^2 \alpha_{22}^2 + \alpha_{12}^2 \alpha_{23}^2 \alpha_{33}^2].
\]

Consider now the case where player 1 is also connected to 3, or the cluster \(\{12, 23, 13\}\) (Figure 6b). Let’s assume constant integration, \(\alpha_{12} = \tilde{\alpha}_{12} + \tilde{\alpha}_{13}\), and \(\alpha_{11}^2 = \tilde{\alpha}_{11}^2\), and where \(\tilde{\alpha}\) indicates a share invested by player 1 in the cluster. Now the variance becomes

\[
\tilde{\sigma}_P^2[\alpha_{r1}] = \sigma_P^2[\alpha_{11}^2 + \tilde{\alpha}_{12}^2 \alpha_{22}^2 + \tilde{\alpha}_{12}^2 \alpha_{23}^2 \alpha_{33}^2 + \tilde{\alpha}_{13}^2 \alpha_{33}^2].
\]

The extra link with node 3 increases the diversification of node 1’s portfolio if and only if \(\sigma_P^2[\alpha_{r1}] \geq \tilde{\sigma}_P^2[\alpha_{r1}]\). Thus,

\[
\alpha_{12}^2 \geq \tilde{\alpha}_{12}^2 + \alpha_{13}^2 \left(\frac{\alpha_{33}^2}{\alpha_{22}^2 + \alpha_{23}^2 \alpha_{33}^2}\right)
\]

(6)
Observe that the extra connection \{13\} always marginally increases the diversification of player 1’s portfolio. This is because \(\alpha^2_{12} > \tilde{\alpha}^2_{12} + \bar{\alpha}^2_{13}\) for any positive \(\alpha \in (0,1)\). However, the diversification decreases with \(\alpha_{23}\). In particular, suppose \(\alpha_{23} = 0\); node 2 is connected to 3 but she does not invest in 3’s assets thus her portfolio return is totally independent of 3’s asset return. In this case the denominator of the fraction in the right side of (6) is the highest possible \((\alpha^2_{22} = \alpha^2_{33})\), and therefore the difference \(\sigma^2_P[r_{\alpha_1}] - \bar{\sigma}^2_P[r_{\alpha_1}]\) is maximal. On the other hand, this difference is minimised when the same denominator is the lowest possible: this happens only when \(\alpha_{23} = 1/(1 + \alpha^2_{33})\).

### A.2 Heterogeneous cost and Core-Periphery structure

One of the stylized facts we observed in the financial market is that it is characterized by a *dissortative* network structure\(^{22}\); few core banks are widely connected while a large number of firms are less connected and mainly to the core ones\(^{23}\). In this section we propose an alternative explanation to the formation of such architectures and their possible evolutions when some of the model parameters change. In particular, we start showing that assuming heterogeneous cost per-link and small enough ambiguity aversion, we can obtain a core-periphery structure.

Suppose a finite set of firms, \(N\), such that \(|N| \geq 3\), and assume for simplicity only two types of players, say \(a\) and \(b\). In particular, \(A \cup B = N\) are the sets of respectively \(a\)-type and \(b\)-type players which differ in their marginal cost per-link: \(c_a < c_b\) for all \(a \in A\) and \(b \in B\). Simplifying the problem even further, assume \(c_a \leq \bar{c}\), the lower bound cost per-link. As before, assume equal shares and constant integration.

**Proposition 3.** Assume \(c_a \leq \bar{c}\), \(c_b > c_a\), and \(\varphi \geq \bar{\varphi}^U\). The network \(G^*\) such that

- \(N_i = N \setminus \{i\}\) for all \(i \in A\)
- \(\delta^+_i > |A| \Rightarrow j \in N_i\) for all \(j \in A\) and \(i \in B\)
- \(\delta^-_i \leq |A| \Rightarrow N_i \subseteq A\) for all \(i \in B\)

\(^{22}\)This architecture characterizes various types of social networks. \cite{Soramäki et al. 2007} in particular shows that the Fedwire bank network structure has a strongly dissortative architecture where few hubs are largely connected while most banks have small number of links. Similar structure is found by \cite{Tabak et al. 2014} for the Brazilian interbank market and \cite{Langfield et al. 2014} for the UK.

\(^{23}\)The role of intermediary node played by the core firms is also analyzed by \cite{Freixas et al. 2000} and \cite{Degryse & Nguyen 2007} among many.
is stable.

**Proof of Proposition 3:** The first point comes directly by the assumption \( c_a \leq c_l \) and definition of \( c_l \). To prove the second point, suppose it was not true, or in a stable network \( G^s, \exists i \in B \) such that \( \delta_i^+ > |A| \) and \( j \in A \) for all \( j \notin N_i \). This means that \( \exists h \in B \) such that \( h \notin N_i \). However, this is not optimal for player \( i \) since \( j \) is more diversified than \( h \), thus \( G^s \) cannot be stable. The proof of the third point follows the same argument.

In other words, the \( a \)-type players are connected to any other node given their minimal marginal cost per-link, while the \( b \)-type players, if connected, will always activate links with \( a \)-type players; for low enough levels of ambiguity aversion, players will activate links to the most diversified nodes given their marginal cost (Figure 7a). When \( c_b \) is high enough, \( b \)-type player could ration her link-portfolio and therefore activate ties only with the core nodes (\( a \)-type).

What may happen to the architecture of this network if \( \varphi \) dropped to levels at most equal to \( \varphi^L \)? First we note that \( a \)-type players will not change their link-structure; the marginal cost is the same and being connected to \((n - 1)\) players, they will not face any uncertainty. On the other hand, \( b \)-type players could change their link-portfolio; For \( c_b \) high enough, they can find profitable to form one or more clique between them: \( i \in N_j \Rightarrow i \in B \ \forall j \in B \) (see Figure 7b).

![Figure 7](image-url)  
**Figure 7:** Core-periphery structure (a) and semi-core-periphery structure (b) with fully connected core and peripheral nodes connected in cliques.

To conclude, under certain conditions we can replicate a core-periphery structure: it is enough to assume risk averse players heterogeneous in marginal cost per-link and with low
enough uncertainty aversion. Under such conditions, the core nodes activate more links due to their relatively low marginal costs and attract more links from the peripheral nodes since they also represent the most diversified alternative. This would not necessarily be true when players are sufficiently ambiguity-averse; the core nodes are still the most diversified players but also the most exposed to uncertain scenarios. This could eventually lead to the formation of cliques among the peripheral nodes.

Part of the existing literature on contagion and financial networks points out about core-periphery structures that the high diversification level of the core banks could help to "stop" contagion dynamics absorbing the defaults of individual peripheral firms. We argue that the robustness of the core could be not fully effective in avoiding global financial distress once we take into account precautionary behavior which may be taken by peripheral banks exposed to the core and thus to more uncertainty.\footnote{Using epidemiology jargon, the robustness of the core firms in terms of high diversification level could be non effective in avoiding "flight" behavior in peripheral banks averse to the uncertainty created by their exposure to the core.}
B Proofs of results

Proof of Proposition 1: We start by characterizing the maximal number of linkages in a out-regular digraph such that the clustering is zero and the minimal cycle length, the girth, is equal to at least four.

Lemma 1. Suppose a digraph $G(N, L)$ of $n \geq 4$ where $\delta_i^+ = k \geq 1$ for all $i \in N$. The structure has zero clusters and girth at least equal to four only if $|L| \leq \frac{n^2}{4}$.

Proof of Lemma 2: We first show that if $|L| \leq \frac{n^2}{4}$ then we can always construct a regular digraph satisfying the conditions. Suppose $|L| = \frac{n^2}{4}$. The regularity in out-degree imposes that if $|L| = \frac{n^2}{4}$, then $\delta_i^+ = \frac{n}{4}$ for all $i \in N$; each node $i$ is at distance one link with other $n/4$ nodes. Suppose a subset of nodes $N_i \subset N$ and the nodes $j \in S_i$, all followers of $i \in N_i$. Similarly we can still find $n/4$ nodes, $q \in S_j$ followers of $j \in N_j \subset N$, which do not belong to the sets $N_i$ and $N_j$, since $|S_j| + |S_i| = n/2$. In words, we can always find followers either of $i$ or $j$ which are all distinct nodes. Similarly, we can still find a set $S_q$ such that $S_q \cap S_i \cap S_j = \emptyset$ since $|S_q| + |S_j| + |S_i| = 3n/4$. Finally, we remain with the nodes in $S_q$ which can always connect with $N_i$, creating a cycle of length 4 which satisfies our constraint on the girth of the graph. The case for $|L| < \frac{n^2}{4}$ follows directly.

We now show that if $|L| > \frac{n^2}{4}$ the condition is never met. Suppose a regular digraph with zero clusters, girth at least equal to four, and $|L| = \frac{n^2}{4} + 1$. Note also that if we find a contradiction for this case, this would be also the case for any higher $|L|$. Using a similar procedure, now $|S_j| + |S_i| = n/2 + 2$ which is more than half of the nodes in $N$. Therefore, it is easy to see that the nodes $q \in S_j$ will connect to $n/4 + 1$ nodes and thus, it must be that $S_q \cap \{S_i \cup S_j\} \neq \emptyset$ since $|S_q| + |S_j| + |S_i| = 3n/4 + 3$; in other words we form either clusters or cycles of length two or three which contradicts the initial assumptions. \hfill \Box

We can now define the architecture of the optimal structure for intermediate cost per-link and for specific ranges of $\varphi \equiv \lambda/\theta$.

First, the optimal structure will be regular since the marginal impact of an extra link, if positive, decreases with the out-degree of the player activating the link; it is optimal to spread the linkages among as many nodes as possible. A central planner would optimally activate an extra link $ij$ for a given cost $c$ if and only if
We will indicate any first order difference due to a link-activation with \( \Delta \), e.g. \( U_i(G + ij, \lambda, \theta) - U_i(G, \lambda, \theta) \equiv \Delta U_i(G, \lambda, \theta) \). We know that the expected return of each portfolio does not change with a link-activation, thus \( \Delta U_i(G, \lambda, \theta) = -\lambda \Delta \sigma^2_P[r_{\alpha_i}] - \theta \Delta \sigma^2_\mu[E[r_{\alpha_i}]] \), and therefore at margin we have

\[
-\lambda \left[ \Delta \sigma^2_P[r_{\alpha_i}] + \sum_{q \in F_i} \Delta \sigma^2_P[r_{\alpha_q}] \right] = c + \theta \left[ \Delta \sigma^2_\mu[E[r_{\alpha_i}]] + \sum_{q \in F_i} \Delta \sigma^2_\mu[E[r_{\alpha_q}]] \right]
\]

(7)

where \( A \) is always negative and \( B \) is positive when \( ij : \{jk\} \in L \land \{ik\} \notin L \), while weakly lower than zero otherwise. Clearly, there always exists a level \( \varphi^U \) such that \(-\lambda A - \theta B \geq 0 \) for a constant \( \lambda > 0 \) and \( \theta \geq 0 \)\(^{25}\). In other words, for \( \theta \) small enough the impact of any new link is positive.

A structure \( G^* \) is optimal if it is the most diversified graph among the set \( G_{n,|L|} \). Suppose \( G^* \) with \( |L| \leq n^2/4 \) active links. We know by Lemma 2 that with such density it is possible to construct a out-regular digraph with no closures. We can show that if such graph has the lowest number of closure is also the most diversified. Suppose an optimal out-regular graph \( G \neq G^* \) with same number of nodes and link-density of \( G^* \). Moreover, assume that in \( G \) either there exists \( i, j \in N : ij, ji \in L \), or \( cl_i(G) > 0 \) for at least one node \( i \). Suppose the former or there is one node \( i \) with one bilateral link. This implies that \( D_{ij}^* \leq D_{ij} \) for all \( i \neq j \) such that \( D_{ij} > 0 \) and \( D_{ij}^* > D_{ij} = 0 \) for one node \( j \neq i \). Thus, \( G^* \) is more diversified than \( G \).

Consider now the second case, or \( i \) forms a closure with other two nodes. Then, there will be an ordered pair \( (i, j) \) in \( G^* \) such that \( D_{ij}^* > D_{ij} = 0 \) and \( D_{ij}^* \leq D_{ij} > 0 \) for all the rest of ordered pairs \((i, j)\), with strict inequality for the one where \( G \) has a clustered triplet. Thus, even in this case \( G^* \) is more diversified than \( G \) and therefore \( G \neq G^* \) cannot be optimal. This also implies that if \( \delta_i^+ = n/4 \) for example, each independent set will be composed by \((n+4)/4\) nodes and thus we could only have at most about \( q = 4n/(n + 4) \) independent sets; more generally, for \( \delta_i^+ \leq n/4 \), a \( q \)-partite graph with \( q \leq \frac{n}{\delta_i^+ + 1} \).

Finally, there always exists a small \( \varphi^L \) such that \(-\lambda A - \theta B \geq 0 \) only if \( B = 0 \). This means that for \( \varphi \leq \varphi^L \), an optimal structure can only be non-ambiguous. Moreover, the unique regu-

\(^{25}\)To quickly check this, assume for example \( \theta \rightarrow 0 \) and thus \( \varphi \rightarrow \infty \). When players becomes essentially ambiguity neutral, \( B \rightarrow 0 \) and thus extra linkages monotonically increase the utility of all the players in \( N \).
lar non-ambiguous digraph is composed by isolated cliques or complete components. Since the diversification of each node increases with her degree, among all the multiple cliques structures, $G^*$ will be the graph with smallest number of components, i.e. largest component’s size $m$, such that $\Delta V(G^*, \lambda, \theta, c) \geq 0$. Moreover, $G^*$ always exists for such intermediary cost $c \in (c, \bar{c})$ since the empty graph $G^0$ can always satisfy $\Delta V(G^0, \lambda, \theta) > 0$. □

**Proof of Proposition 2:** We start showing that any stable structure must be out-regular, or $\delta^+_i = k \geq 0$ $\forall i \in N$. We start assuming $\varphi \geq \bar{\varphi}^U$ where $\bar{\varphi}^U$ is the ratio such that $-\lambda\Delta\sigma^2_P[r_{ai}] - \theta\Delta\sigma^2_{\mu}[E[r_{ai}]] = 0$ for any new link-activation; for such levels of $\varphi$, the positive diversification effect due to the new link activated is weakly greater than any negative effect due to higher exposure to uncertainty. Suppose an out-regular graph $G$ which is non-stable. Then, consider the node $i \in N$ with the highest out-degree, which is $i$ such that $\delta^+_i \geq \delta^+_j \ \forall j \neq i$. Suppose a node $j \in N$ which does not share neighbours with $i$, i.e. $N_i \cap N_j = \emptyset$. Then it must be that $\delta^+_i = \delta^+_j$ since to get the same benefit from her (completely) distinct link structure, $j$ must have an equivalent one. Consider now a distinct node $h$ which shares at least one neighbour with $i$ ($N_i \cap N_h \neq \emptyset$). First note that $hi \in L$ since $h$ can partially replicate the portfolio diversification of $i$ by investing directly in it. This implies that $hi, ih \in L \ \forall h \in N_i$. Moreover, if $i$ finds it optimal to keep such a link structure despite the bilateral ties, it must be that $\delta^+_h = \delta^+_i$ since they can always optimally replicate the neighbourhood of $i$. Hence, the graph must be out-regular. We will see later that the regularity for the case $\varphi \leq \bar{\varphi}^L$ is a direct consequence of the equilibrium condition.

Suppose again $\varphi \geq \bar{\varphi}^U$. Consider $c \in [c', \bar{c}]$ and suppose a regular $G^s$ which is not $q$-partite. Then it must exists at least one cluster in $G^s$. Define $ih \in L$ as one of the links of such cluster. Then $G^s$ is not stable by definition of $c'$, there must exists at least another node $j$ such that $ij \notin L$ and such that $\tilde{cl}(G^s) = \tilde{cl}(G^*)$ with $\tilde{G}^*$ the network $G^*$ with $ih$ replaced by $ij$, i.e. there exists a profitable deviation for node $i$. Therefore, $G^s$ must be $q$-partite.

Observe that there exists $\theta \in \mathbb{R}^+$ such that $\Delta U_i(G + ij) = 0 \ \forall ij$ such that $\tilde{cl}(G + ij) = \tilde{cl}(G) \land \delta^+_j > 0$. Thus, for a constant $\lambda$, define such a level as the lower bound $\bar{\varphi}^L$. Moreover, note that $\bar{\varphi}^L$ is such that $-\lambda\Delta\sigma^2_P[r_{ai}] - \theta\Delta\sigma^2_{\mu}[E[r_{ai}]] = 0$ for a constant $\lambda$ and $\theta \geq 0$ only if $\Delta\sigma^2_{\mu}[E[r_{ai}]] = 0$. Assume $\varphi \leq \bar{\varphi}^L$. We know that, by construction, the intermediary cost $c \in (c', \bar{c}]$ excludes that $G^s = K^n$, i.e. the stable network cannot be complete. However, it allows for smaller non-connected cliques; if $c = c^u$, $G^s$ will be composed by $q = n/2$ components (dyads) where $\delta^+_i = 1$ for all $i \in N$, while for $c < c^u$ the size of the possible stable connected components in $G^s$ weakly increases. □

\footnote{A complete network structure is never optimal if we assume intermediary cost $c \in (c, \bar{c})$.}
Proof of Corollary 1: A stable network is optimal only if the net impact of a linkage activated by a node $i$ on his parent nodes $j \in S_i(G)$ is equal to zero. It is easy to see that each player does not always internalize the impact of her link-choice on the rest of her peers. Each individual agent $i$ optimally activates a link $ij$ until

$$-\lambda \Delta \sigma^2_P[r_{\alpha_i}] - \theta \Delta \sigma^2_{\mu}[E[r_{\alpha_i}]] = c$$

Define the aggregate impact of an extra link $ij \notin L$ on $q \in S_i$ as

$$C \equiv -\lambda \sum_{q \in F_i} \Delta \sigma^2_P[r_{\alpha_q}] - \theta \sum_{q \in F_i} \Delta \sigma^2_{\mu}[E[r_{\alpha_q}]]$$

It is clear that if $\lambda \neq 0$ and either $\theta = 0$ or $\Delta \sigma^2_{\mu}[E[r_{\alpha_i}]] \leq 0$ for any extra link, then comparing [7] to [8] we note that $G^s \neq G^*$ since $C > 0$. In particular, since each agent $i$ does not take into account the positive impact of the link activation on any $j \in S_i$, $G^s$ can only be less dense than $G^*$. On the other hand, suppose $\theta > 0$ and that for any extra link not in $L$ we have $\Delta \sigma^2_{\mu}[E[r_{\alpha_i}]] > 0$. If $C = 0$, then $G^s = G^*$ since the net impact of any extra link is equal to zero and thus the optimality conditions of the central planner and of each individual agent coincide. Suppose $C > 0$. Then there exists at least one extra link such that activating it, the uncertainty as perceived by at least some of the $q \in S_i$ increases. In particular, $-\lambda \sum_{q \in F_i} \Delta \sigma^2_P[r_{\alpha_q}] > \theta \sum_{q \in F_i} \Delta \sigma^2_{\mu}[E[r_{\alpha_q}]]$, or the stable $G^s$ must be less dense than $G^*$ since $i$ would not take into account the net positive effect on other peers. Finally if $C < 0$, then $-\lambda \sum_{q \in F_i} \Delta \sigma^2_P[r_{\alpha_q}] < \theta \sum_{q \in F_i} \Delta \sigma^2_{\mu}[E[r_{\alpha_q}]]$, or $G^s$ must be more dense than $G^*$, since each node $i$ fails to internalize the net negative impact due to extra uncertainty for the rest of her peers.

$\square$