Choice under Risk and Uncertainty (Part I)

Week 10

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Outline

- Introduction
- Risky outcome
- Choice under Uncertainty
  - St. Petersburg Paradox
  - Von-Neumann and Morgenstern Utility function
  - The axioms of rational choice
- Risk Aversion
- Mean-Variance Choice
- Insurance

Introduction

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- Probability distribution
- Expected value
- Variance

More formal notation

- We assume a finite set of outcomes $\chi = \{x_1, ..., x_n\}$
- A lottery over χ is a vector $p = (p_1, ..., p_n)$ with p_i the probability that outcome x_i occurs.
- The set of lotteries over χ is then $\mathcal{P} = \Delta(\chi)$.
- Given two lotteries p and p', any convex combination $\alpha p + (1 - \alpha)p'$ with $\alpha \in [0, 1]$ is also a lottery (*compound lottery*).

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How much money do you expect to make?

What is the *expected value* of your investment?

The expected value of a lottery is the average payoff that the lottery will generate:

$Expected\ value = Prob.\ of\ A \times Payoff\ if\ A\ occurs\ + Prob.\ of\ B \times Payoff\ if\ B\ occurs + Prob.\ of\ C \times Payoff\ if\ C\ occurs$

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In our example,

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- $\$120$ with probability 0.3.
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$$EV = (0.3 \times 120) + (0.4 \times 100) + (0.3 \times 80) = 100$$

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The two investments have the same expected value ($\$100$).

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However, the investment on Internet company is more *risky* than the one in Public utility: The first has greater likelihood of going up or down.

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How do we compute the variance?

- Compute the expected value (in this case 100).
- Compute the squared deviation of each outcome from the expected value:

$$\begin{aligned} (120 - EV)^2 &= (120 - 100)^2 = 400 \\ (100 - EV)^2 &= (100 - 100)^2 = 0 \\ (80 - EV)^2 &= (80 - 100)^2 = 400 \end{aligned}$$

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For the investment on Public utility, deviations are the same, but the variance is smaller!

$$\text{Variance} = (0.1 \times 400) + (0.8 \times 0) + (0.1 \times 400) = 80$$

Choice under uncertainty

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Example:

- Lottery A: pays 20 with probability 0.8 and 200 with probability 0.2.
- Lottery B: pays 4 with probability 0.9 and 400 with probability 0.1.

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Do we use the expected value to assess its attractiveness?

St. Petersburg Paradox

Problem posed by Bernoulli in 1728.

Suppose someone offers to toss a fair coin. The pot starts at 2 dollars and is doubled every time a head appears. The first time a tail appears, the game ends and the player wins whatever is in the pot. In short, the player wins 2^k dollars, where k equals number of tosses.

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The expected value of this gamble is infinite!

$$\frac{1}{2} \cdot 2 + \frac{1}{4} \cdot 4 + \frac{1}{8} \cdot 8 + ... + \frac{1}{2^k} \cdot 2^k + ... = \infty$$

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$$\text{B) } 0.9 \cdot u(4) + 0.1 \cdot u(400)$$

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Von-Neumann and Morgenstern utility function.

VnM utility function

Definition

A utility function $U : \mathcal{P} \to \mathbb{R}$ has an **expected utility form** (or is VnM utility function) if there are numbers $u_1, ..., u_n$) for each of the N outcomes $(x_1, ..., x_n)$ such that for every $p \in \mathcal{P}$, $U(p) = \sum_{i=1}^{n} p_i u_i$.

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Theorem

(VnM 1947) A complete and transitive preference relation $\succeq$ on $\mathcal{P}$ satisfies continuity and independence if and only if it admits a expected utility representation $U : \mathcal{P} \to \mathbb{R}$.

Notice that the (VnM) expected utility is the expected value of utilities over prizes. It has two important features:

- It is additive over states of the world.
- It is linear in probabilities

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What do we (implicitly) assume about rational choice in order to obtain these features?

The axioms of rational choice

Two main assumptions:

- Continuity
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The first axiom requires that the preference relation on the space of lotteries is continuous: small changes in probability should not change the nature of the ordering between two lotteries.

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- Independence

The second axiom is also known as the axiom of complex gambles. It roughly says that two gambles mixed with a third one maintain the same preference order as when the two are presented independently of the third one.

Continuity

Suppose that I prefer a sunny weekend in Barcelona to a rainy weekend in London. Continuity of preferences requires that I should also prefer a weekend in Barcelona when there is a small probability of rain, to a weekend in London.

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Is it always true? Lexicographic preferences?

Independence

If $x \sim y$, then $\forall z, p$

$$xp + z(1-p) \sim yp + z(1-p)$$

Similarly, if $x \succ y$, then $\forall z, p$

$$xp + z(1-p) \succ yp + z(1-p)$$

Independence

Suppose that you face two gambles. In one you can win with probability 0.99 a trip to Paris and with probability 0.01 a movie about Paris. In the other one, you can still win a trip to Paris with probability 0.99 but with probability 0.01 you simply stay home. Suppose that you rank Trip to Paris $\succ$ Movie $\succ$ Home.

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The Independence axiom requires that since Movie $\succ$ Home, then

$$\text{Trip}0.99 + \text{Movie}0.01 \succ \text{Trip}0.99 + \text{Home}0.01$$

Independence

Is this always reasonable? What if you are so disappointed that you are not going to Paris that you cannot face the idea of watching a movie about Paris!? (**Machina Paradox**)

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The independence axiom does NOT allow for regret, which could well affect how people make risky choices.

Allais paradox

You are asked to choose between the following two gambles:

Gamble A. A 100% chance of receiving 1 million.

Gamble B. A 10% chance of receiving 5 million, an 89% chance of 1 million, and a 1% of receiving nothing.

Allais paradox

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Gamble A. A 100% chance of receiving 1 million.

Gamble B. A 10% chance of receiving 5 million, an 89% chance of 1 million, and a 1% of receiving nothing.

Pick one of this gamble and write it down.

Allais paradox (cont.)

Gamble C. An 11% chance of receiving 1 million, and 89% chance of nothing.

Gamble D. A 10% chance of receiving 5 million, an 90% chance of receiving nothing.

Allais paradox (cont.)

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$$A \succeq B \Rightarrow u(1) > 0.1u(5) + 0.89u(1) + 0.01u(0)$$

rearranging we obtain

$$0.11u(1) > 0.1u(5) + 0.01u(0)$$

Allais paradox (cont.)

Adding $0.89u(0)$ to each side we obtain

$$0.11u(1) + 0.89u(0) > 0.1u(5) + 0.90u(0)$$

or that $C \succ D$ by an expected utility maximizer!

Risk aversion

Simple choice problem.

Consumer has 10 of wealth and is contemplating a gamble that will give him a probability 50% of winning 5 and 50%, of losing 5.

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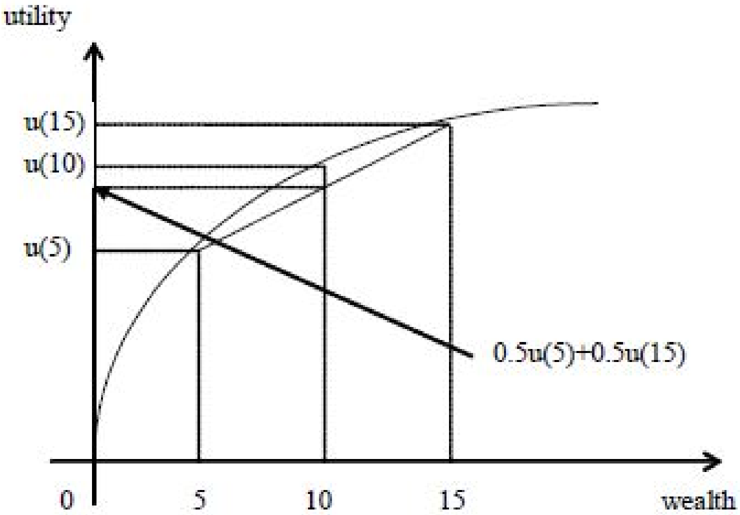
The expected utility is $(0.5) \cdot u(5) + (0.5) \cdot u(15)$.

Risk aversion



Risk aversion

Note that the expected utility of wealth is less than the utility of the expected wealth.

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$$u\left(0.5 \cdot 5 + 0.5 \cdot 15\right) > 0.5 \cdot u(5) + 0.5 \cdot u(15)$$

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The consumer is **risk-averse**: he prefers to have the certain amount of money equal to the expected value of his gamble (10) rather than face the gamble.

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The consumer is **risk-averse**: he prefers to have the certain amount of money equal to the expected value of his gamble (10) rather than face the gamble.

Or alternatively said, the **risk-averse** consumer has **concave** utility function.

A consumer who has **convex** utility function is **risk-lover**.

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A **linear** utility function corresponds to a **risk-neutral** consumer.

Example

You are about to graduate and you have two job offers:

1. Join a large established company at an income of $\$35K$ per year.
2. Join a new start-up company at a token salary of $\$5K$ per year but with the possibility of earning of bonus $\$60K$ at the end of the year, in case of success. You estimate the probability of success to be equal to 50%.

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The expected value of your salary is the same. Are you indifferent between the two alternatives?

Example

Your preferences over lotteries are such that you rank them NOT according to their expected value, BUT according to the expected utility that they generate for you.

$$\text{EV} = \prod_i \alpha_i \pi_i$$
$$\text{EU} = \prod_i \alpha_i u(\pi_i)$$

with α_i defining the probability of event i and π_i the payoff or outcome of event i.

Example

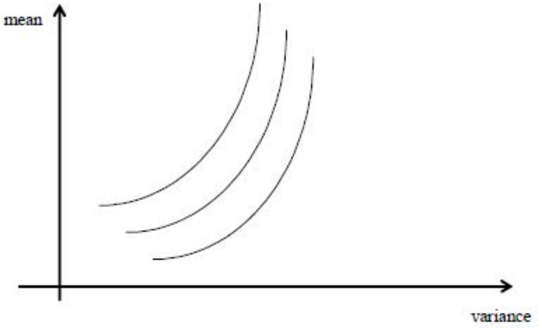
If you are risk-averse, between the two alternatives with the same expected value, you prefer the alternative with lower variance.

If you are risk-neutral, you are indifferent between any two alternatives with the same expected value.

If you are risk-lover, between two alternatives with the same expected value, you prefer the one with higher variance.

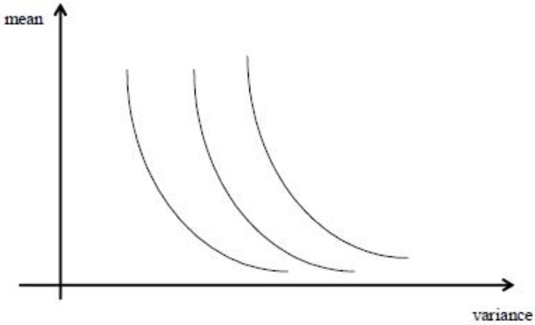
Mean-Variance Choice

We can represent indifference curves in a mean-variance space for a **risk-averse** individual



Mean-Variance Choice

For a **risk-lover** individual



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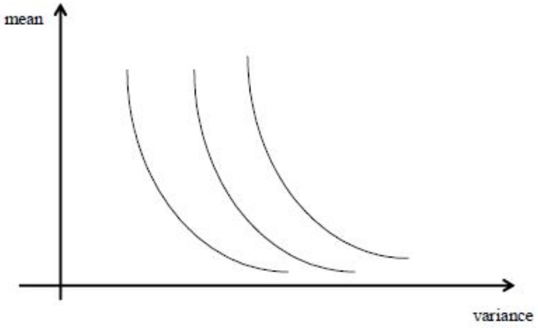

What about the indifference curves in a mean-variance space for a **risk-neutral** consumer?

Certainty equivalent

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Example?

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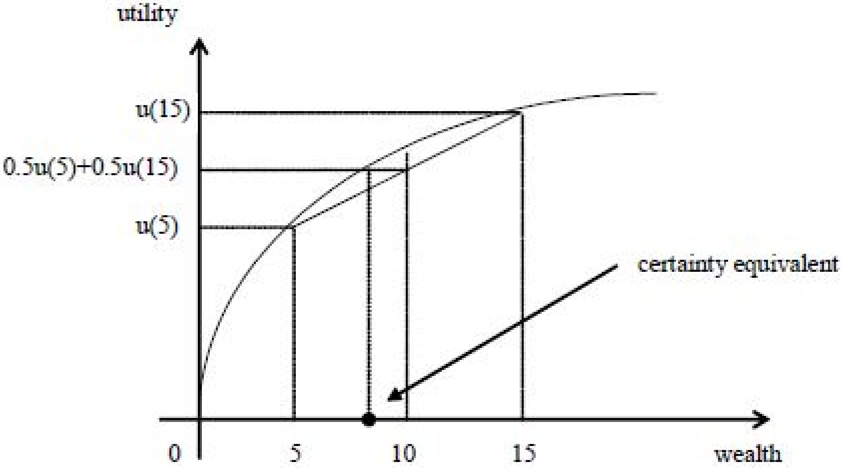
Example?Safe job-Risky job, Bond-Risky asset, etc...

Certainty equivalent



Insurance

Why we are keen to pay a **premium** for sure to be covered for a loss that will only occur with some probability?

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What is the (highest) premium that we are happy to pay?

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Insurance (Example)

A consumer has risky income Y such that

- $Y = 1$ in the "bad" state, with probability 0.5,
- $Y = 3$ in the "good" state, with probability 0.5.

What is the consumer's expected income?

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What is the consumer's expected income?

$$EI = 0.5 \cdot 1 + 0.5 \cdot 3 = 2$$

Insurance (Example)

Assume that $U(c)$*, with* $C = Y$ *in both states, is concave. Show geometrically that the consumer's expected utility will be less than the utility he would receive from a constant level of* C *equal to* $E(C) = E(Y)$.

Insurance (Example)

Show geometrically that the consumer will be willing to pay a "risk premium" in terms of lost income in both states, if consumption were equal in both states.

Insurance (Example)

Give an algebraic expression which implicitly defines this risk premium.

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$$U[E(Y) - p] = E[U(Y)]$$

Insurance (Example)

Show that if the consumer can buy insurance at "fair" premia, he will fully ensure himself against fluctuations in their income.

Fair insurance when insurance company makes zero expected profits:

Fair premium : $0.5 \cdot p2 + 0.5 \cdot p2 - 0.5 \cdot 2 = 0 \rightarrow p = 0.5$

Insurance (Example)

$$\text{Fair premium}: p = 0.5$$

If the consumer buys full insurance he will get the same income in both states,

$$3 - 2 \cdot 0.5 = 1 + 2 - 0.5 \cdot 2$$

Insurance (Example)

The expected utility if insured will be $E[U^I] = U(3 - 2 \cdot 0.5) = U(2)$, while the expected utility if NOT insured is $E[U^{NI}] = 0.5U(3) + 0.5U(1)$.

Insurance (Example)

The expected utility if insured will be $E[U^I] = U(3 - 2 \cdot 0.5) = U(2)$, while the expected utility if NOT insured is $E[U^{NI}] = 0.5U(3) + 0.5U(1)$.

But we know that $U(2) > 0.5U(3) + 0.5U(1)$, thus the consumer will get the full insurance!

Insurance (Example)

Give two reasons why an insurance company is unlikely to offer "fair" premia.

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Give two reasons why an insurance company is unlikely to offer "fair" premia.

- Not perfectly competitive market (market power)
- Asymmetric information: the insurance company does not know characteristics of consumer and often the consumer can directly influence his income.