Microeconomic Analysis

Optimization "Cookbook" and Demand function

Marco Pelliccia (mp63@soas.ac.uk, Room 474)

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Outline

1. Utility Maximization Problem (UMP)
   - Basic Problem
   - Constrained optimization
   - Lagrangian method
   - Envelope Theorem
2. Expenditure Minimization Problem (EMP)
   - Basic Problem
   - Duality
   - Hicksian (Compensated) Demand

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What is the consumer's problem? (in words) Throughout, we will assume:

- The preference relation, $\succeq$, is rational, continuous, and locally non-satiated.
- $x \in X = \mathbb{R}^N_+$
- $u(x)$ is a continuous utility function representing $\succeq$.

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Utility Maximisation Problem

For positive price p and wealth w, the consumer solves

$$\max_{x \in X} u(x)$$
$$s.t.\ px \leq w$$

However, given the locally non-satiation assumption, we can reduce the constraint set to $px = w$.

Lagrangian method

Generalising the problem we can say that we want to solve the following problem

$$\max_{x \in X} f(x) \quad \text{s.t.} \quad g(x) = c$$

with $f(x)$ and $g(x)$ two C_1 functions, c a real number, and $x = (x_1, ..., x_n)$ a vector.

We remark that $g(x) = c$ does NOT have to be the unique constraint!

Lagrangian method

- Set up the Lagrangian:

$$\mathcal{L}(x, \lambda) = f(x) + \lambda(c - g(x))$$

- Set up the FOC:

$$\frac{\partial \mathcal{L}}{x} = 0$$

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Lagrangian method

Note that we can impose the partial derivatives $\frac{\partial \mathcal{L}}{x} = 0$ assuming that $x^* > 0$, or that the solution will be **interior**. Otherwise, we need to set up the K-T FOC.

Example

$$\max_{x_1,x_2} u(x_1, x_2) = 2x_1 x_2^{1/2}$$

$$\text{s.t. } p_1 x_1 + p_2 x_2 = w$$

Example

- Set up the Lagrangian:

$$\mathcal{L}(x_1, x_2, \lambda) = 2x_1 x_2^{1/2} + \lambda(w - p_1 x_1 - p_2 x_2)$$

To simplify our computation, we can transform the utility function as $\ln u(x_1, x_2) = \ln 2 + \ln x_1 + (1/2) \ln x_2$.

Example

- Set up the FOC:

$$\frac{\partial \mathcal{L}}{x_1} = 0 \Rightarrow \frac{1}{x_1} - \lambda p_1 = 0$$
$$\frac{\partial \mathcal{L}}{x_2} = 0 \Rightarrow \frac{1}{2x_2} - \lambda p_2 = 0$$
$$\frac{\partial \mathcal{L}}{\lambda} = 0 \Rightarrow w - p_1 x_1 - p_2 x_2 = 0$$

Solving the system we obtain $x_1^* = \frac{2w}{3p_1}$, $x_2^* = \frac{w}{3p_2}$.

Envelope Theorem

Suppose again the constrained optimization problem

$$\max_{x \in X} f(x;q)$$
$$\text{s.t.} \qquad g_j(x;q) = c$$

where x is a n-vector of variables, f and g_j with $j = \{1, ..., m\}$ are continuously differentiable functions, and q is a k-vector of parameters. Assume that for each q, there exists a unique solution $x^*(q)$.

Envelope Theorem

If we evaluate the function $f(x, q)$ at its optimal x^* for a given q we obtain the *value function*,

$$v(q) = f(x^*(q); q)$$

We can ask **how does** $v(q)$ **move with elements of** q **around some reference level** $\bar{q}$ **?.**

Envelope Theorem

So, we want to calculate $v'(q)_h$ with $h = 1, ..., k$. If $x^*(q)$ is differentiable we can use the chain rule to derive

$$v'(q)_h = \sum_{i=1}^{n} f_i'(x^*(q), q)\frac{\partial x^*}{\partial q_h}(q) + f_h'(x^*(q), q)$$

Then if the Jacobian matrix of the constraint has rank m, there exists a unique vector $\lambda_1(q), ..., \lambda_m(q)$ for each q such that the solution $x^*(q)$ satisfies the FOC

$$f_i'(x^*(q), q) - \sum_{j=1}^{m} \lambda_j(q)\frac{\partial g_j}{\partial x_i}(x^*, q) = 0 \ \text{ for } i = 1, ..., n$$

Thus we have,

$$v'(q)_h = \sum_{i=1}^{n} \left[\sum_{j=1}^{m} \lambda_j(q) \frac{\partial g_j}{\partial x_i}(x^*, q) \right] \frac{\partial x^*}{\partial q_h}(q) + f'_h(x^*(q), q)$$

Reversing the sum we obtain

$$v'(q)_h = \sum_{j=1}^{m} \lambda_j(q) \left[\sum_{i=1}^{n} \frac{\partial g_j}{\partial x_i}(x^*, q) \frac{\partial x^*}{\partial q_h}(q) \right] + f'_h(x^*(q), q)$$

Envelope Theorem

we know that for any $j = 1, ..., m$ it must be that $g_j(x^*(q), q) = 0$ for all q. Thus, if we differentiate this equation with respect to q we obtain,

$$\frac{\partial g_j}{\partial x_i}(x^*(q), q)\frac{\partial x_i^*}{\partial q_h}(q) + \frac{\partial g_j}{\partial q_h}(x^*(q), q) = 0$$

Hence,

$$v'(q)_h = -\sum_{j=1}^{m} \lambda_j(q) \left[\frac{\partial g_j}{\partial q_h}(x^*(q), q)\right] + f_h'(x^*(q), q)$$

Envelope Theorem

Recall that the Lagrangian of $f(x, q)$ is

$$\mathcal{L}(x, q) = f(x, q) - \sum_{j=1}^{m} \lambda_j g_j(x, q)$$

Therefore, we have

$$\mathcal{L}'_h(x^*(q), q) = v'(q)_h \quad \text{for } h = 1, ...k$$

Example

Consider the UMP problem,

$$\begin{aligned} \max_{x \in X} u(x) & \\ \text{s.t.} \qquad & px = w \end{aligned}$$

where x is a bundle of goods, p is the relative price vector, and w is the consumer's wealth. Denote the solution to the problem with $x^*(p, w)$. Then we obtain the value function,

$$v(p, w) = u(x^*(p, w))$$

for every p and w, which is called the **Indirect Utility function.**

Example

Note that the utility function does not depend directly on p and w. Thus, applying the Envelope Theorem we have,

$$v_i'(p,w) = -\lambda^*(p,w)x_i^*(p,w)$$

and

$$\frac{\partial v}{\partial w}(p,w) = \lambda^*(p,w)$$

Thus,

$$-\frac{\frac{\partial v}{\partial p_i}(p,w)}{\frac{\partial v}{\partial w}(p,w)} = x_i^*(p,w)$$

That is, if you know the Indirect Utility function, you can recover the demand functions. This result is known as **Roy's Identity**.

We can always solve constrained *minimization* problem following the same steps proposed for the constrained maximization case. For a consumer, the EMP is

$$\begin{array}{cc} \min_{x \in X} px & \\ \text{s.t.} & u(x) = \bar{u} \end{array}$$

The Lagrangian of this problem will be

$$\mathcal{L}(x; p, \bar{u}) = px - \lambda(u(x) - \bar{u})$$

The solution to the EMP problem is the vector $h^*(p, \bar{u})$ of **Hicksian demand functions.**

- $h^*(p, \bar{u})$ specifies a consumption bundle for a given price vector p and utility level $\bar{u}$.
- It answers the question "*Holding the level of utility constant at $\bar{u}$, how does the consumer change the consumption bundle after a change of price?*".
- $e(p, \bar{u}) = ph^*(p, \bar{u})$ is the **Expenditure function**, or the value function of the EMP problem.

EMP

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$$EMP = UMP$$

This result is defined as ***Duality***

Relationship between Marshallian and Hicksian demand function

Proposition

Suppose $u(.)$ is a continuous utility function representing a locally non-satiated strictly convex preference relation $\succeq$ defined on the consumption set $X = \mathbb{R}^N_+$. For all p and $\bar{u}$, the Hicksian demand $h(p, \bar{u})$ is the gradient of the expenditure function with respect to prices:

$$h(p, \bar{u}) = \nabla_p e(p, \bar{u})$$

That is, $h_i(p, \bar{u}) = \partial e(p, \bar{u}) / \partial p_i$ for all $i = 1, ..., N$.

Relationship between Marshallian and Hicksian demand function

Proposition

*(**Slutsky Equation**) Suppose $u(.)$ is a continuous utility function representing a locally non-satiated strictly convex preference relation $\succeq$ defined on the consumption set $X = \mathbb{R}^N_+$. Then for all (p, w) and $u = v(p, w)$ we have*

$$\frac{\partial h_l}{\partial p_k}(p, \bar{u}) = \frac{\partial x_l}{\partial p_k}(p, w) + \frac{\partial x_l}{\partial w}(p, w) x_k(p, w)$$

for all l and k.

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- $x(p, w) = h(p, v(p, w))$
- $h(p, \bar{u}) = x(p, e(p, \bar{u}))$
- $e(p, \bar{u}) = v(p, e(p, \bar{u}))$
- $v(p, e(p, \bar{u})) = e(p, v(p, w))$
- $h(p, \bar{u}) = \nabla_p e(p, \bar{u})$
- $-\frac{\frac{\partial v}{\partial p_i}(p,w)}{\frac{\partial v}{\partial w}(p,w)} = x_i^*(p, w)$ (*Roy's Identity*)
- $\frac{\partial h_l}{\partial p_k}(p, \bar{u}) = \frac{\partial x_l}{\partial p_k}(p, w) + \frac{\partial x_l}{\partial w}(p, w)x_k(p, w)$ (*Slutsky Equation*)

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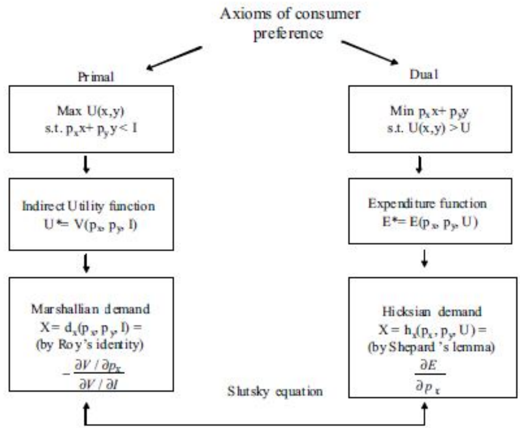

Figure: Summary