Risk-sharing and Probabilistic Network Structure

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Abstract

We study the impact of a risk-sharing network structure on the risk-exposure of its players. We show that, even assuming homogeneous risk attitudes, the players' optimal risk-choice could vary once we assume frictions on the flow of transfers between agents. In particular, we consider a two-period model and assume a *probabilistic network structure*; each link existing at time t could decay at $t+1$ for some i.i.d. homogeneous probability $\delta \in (0,1)$. Moreover, assuming positive degrees of risk-correlation, each link becomes a channel of both risk and liquidity, and a location which benefits more in terms of risk-pooling could become too exposed to risk for high correlation levels. We also indicate that individual optimal risk-exposures do not necessarily coincide with socially optimal risk-exposures. This can occur under certain parameter specifications due to the negative externality created by each individual's optimal choice, as she does not internalize the social cost of her action on the rest of the players. Finally, our findings suggest that, all things being equal, a *complete* network could be the optimal structure in terms of risk-pooling degree and ability to "absorb" the systemic risk produced by the whole risk-sharing component. More generally, a structure characterized by *triadic closures* combines these two features.

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1 Introduction

The theoretical literature on informal risk-sharing is extensive. In contexts where there are no formal markets for insurance, such as in villages in developing countries, households create informal revenue sharing schemes to offset periodic income fluctuations (Udry [1995] and Townsend [1995]). As stressed by this literature, most of the channels for informal risk-sharing coincide with specific social networks composed by relatives or friends. Moreover, although the *global risk-sharing* hypothesis is rejected by data, given a system of interconnected agents, the possibility of indirect flows of transfers between individuals cannot be excluded, particularly within certain ranges of social distance between peers.[1]

We study the interaction between risk taking, risk sharing and network structure. In particular, we analyse the problem faced by a finite set of identical agents[2], connected in a risk-sharing network, optimally allocating capital shares between two assets, a *risky* and a *risk-free* one. There are only two periods: at t, each agent decides his portfolio composition which will pay returns at $t+1$. Players are assumed to agree to an exogenous *egalitarian* risk-sharing norm: at $t+1$, the total income collected by all the agents in a component is split equally between them. There is no strategic deviation from this sharing norm, therefore we assume that each agent can receive or send liquidity transfers at $t+1$ from or to any player of the same component through a chain of linkages connecting them.[3] Under these conditions, players' risk-exposure choices would not be conditioned by their location in the network.

Following on from this, we assume exogenous frictions on the transfer flow between two peers; any link which exists at time t may "decay" at $t+1$ with a fixed and independent probability $\delta \in (0,1)$.[4] As a result, the location of each agent in the risk-sharing network may now differentiate the players' expected flow of liquidity from or to the rest of the component (risk-pooling degree), and consequently affect their final optimal portfolio choice.

Furthermore, we analyse the impact of positive correlated risk among the risky assets, expressed by the correlation coefficient $\varphi \in (0,1]$, on the players' risk exposures. We show that the final equilibrium configuration, defined by the vector of optimal capital shares invested

[1] Townsend [1995], Bramoullé and Kranton [2007] and Ambrus et al. [2014] pointed out that for an arbitrarily large number of bilateral interactions, even in the absence of a cohesive risk-sharing group, bilateral risk-sharing relations could end up extending the risk-sharing norm to the whole component via intermediary pair-wise relations.

[2] Homogeneous in risk-attitude and capital endowment.

[3] We assume that the players do not strategically *free-ride* on the choices of the rest of the peers.

[4] We do not consider any activation of new linkages in this chapter. This restriction could be reasonable in environments where creating a new connection requires more time or cost than severing an existing one. δ can equally be seen as a probability coefficient or a generic discount factor. Any link formation dynamic, strategic or not, would open new research questions not specifically addressed by this paper.

in risky-assets, **x**, synthesizes the trade-off between the *centrality* of a location in the network and the *criticality* of this location with respect to systemic-risk exposure; for relatively low correlation levels, the agents who benefit the most are those in locations which guarantee liquidity with relatively higher probability (high expected risk-pooling), while for relatively high correlation levels, the same agents could become more exposed to systemic risk.

A wide variety of models which analyse the strategic choices of agents connected in a network structure consistently show that individual choices do not take their impact on the overall system into account.[5] When we consider positive asset correlation, the volatility of each agent's portfolio becomes a partial or complete substitute for the risk taken by other peers in the component. An interesting result of this is that, due to this misalignment of incentives, the individual optimal risk-exposures will not necessarily coincide with the level maximizing an utilitarian social welfare function (SWF); under certain conditions the structural frictions on the transfers between agents and positive correlation degrees could lead to inefficient individual risk-exposures.

The relative risk-pooling benefit faced by a player i is function among other things of the number of players with which he expects to risk-share in the following period. This number is computed for each player i by a specific measure, n_{it}^{-1}. In contrast to most of the standard network centrality measures in the literature, it scores maximal value when the structure is a *complete graph* and lowest value when i is a peripheral node in a line graph. We show that in our context, n_{it}^{-1} catches the positive and negative impacts of a network structure on the portfolio composition of a node i; for positive correlation coefficients, each connection simultaneously becomes a channel of both risk and liquidity, and the trade-off between these two characteristics ranks each player's location in the network.

The closest work to ours is Belhaj and Deroïan [2012]. The authors analyze an informal risk-sharing network in which players agreed to a bilateral partial sharing rule. Our setting differs in that we study the implications of a probabilistic network structure (structural frictions) on the optimal risk-exposures of the agents. This structural uncertainty will partially differentiate our findings from theirs.[6] Bramoullé and Kranton [2007] analyze the behavior of agents in a risk-sharing network as function of their location in the structure. Our paper differs in at least three aspects. First, one of their theoretical contributions focuses on the link-formation process, an aspect which we do not address in our model. Second, they assume bilateral risk-sharing agreements while we study a sharing rule at component level.

[5] See Bramoulle et al. [2016] for an extensive review of the literature.

[6] This is particularly clear if we look the conclusion in Section 4 of their work, when they allow for group-sharing rules.

Third, we study the impact of a non-negative degree of correlation among the risky assets, an aspect which they do not consider in their analysis. In line with Krishnan and Sciubba [2009], our model highlights the role of the network architecture on economic performances. In particular, we share the view that *social cohesion* could improve the risk-sharing of a community. However, under our model setting the minimum degree of diversification necessary to see positive risk-sharing effects is function, among other things, of the positive correlation parameter φ. For important affinities, we would like to mention Ambrus et al. [2014]. In this work the authors model the theoretical conditions necessary to observe "full" informal risk-sharing networks and the formation of risk-sharing "islands". We share their interpretation of a *risk-sharing arrangement* but, as previously stated, we substitute their *incentive compatibility* assumption with the probabilistic characteristic of the network structure. Moreover, the authors assume *full information*, or perfect monitoring between peers. While this assumption could be reasonable at village level (Udry [1995]), it could be restrictive for other applications such as financial networks. We argue that a probabilistic structure could be interpreted as a way of modelling agents with limited ability to monitor their partners. Gottardi et al. [2014] analyze the capacity of a risk-sharing system to absorb shock as function of the specific patterns interconnecting financial firms. We share many elements with their model, particularly the fact that we both model systemic risk as mainly driven by correlation in agents' asset portfolios[7]. However, they model the possibility of a firm's default with the relative contagion effects on the interconnected partners due to an exogenous and random shock hitting the asset owned by the firm. In our model, we focus on the agents' optimal portfolio composition, and the mechanism of risk transmission between firms makes use of the risk-sharing norm which is arranged *ex-ante*. This and other differences in our models' setting lead to different conclusions; among others, in **?** a completely connected component is optimal only for relatively small shocks, while in our model the complete network is always optimal.[8]

Among the empirical papers studying the trade of liquidity between financial institutions we highlight Pozzi et al. [2013]. These authors in particular represent through a graph the complex dependency structure in the financial equities market. They found that stocks which are more peripheral in such a network structure perform better in terms of lower volatility and higher returns. They note that this could suggest that the central stocks, according to

[7] Elsinger et al. [2006] propose an empiric analysis using Austrian data aiming to show this specific point.

[8] The difference of this specific conclusion is mainly due to the fact that in our model the agents adjust their portfolio composition to offset the risk taken by their partners, when we assume positive correlation among their assets. In other words, the network does not necessarily describe the correlation structure itself as in **?**. In our model the correlated risk impacts all the players of a connected component to an extent, function of the relative players' location.

an aggregate centrality measure, perform poorly because of their high exposure to sudden market perturbation and the strong herd effect they are exposed to; we remark that we could observe a similar result in our model for relatively high φ values. Finally, Li and Schürhoff [2012] present empirical findings which are particularly close to our model predictions. In particular, the authors study a trade market of bonds using the MSRB Transaction Reporting System. They report that the more central dealers in such a network hold more volatile portfolios than peripheral ones due to their relative location advantage. A similar result is obtained in our chapter for relatively low correlation levels.

We believe that this paper can improve the literature on risk-sharing by highlighting the impact of structural frictions and positive correlated risk on the risk-exposures chosen by agents in a risk-sharing community. In particular, given the generally non-monotonic impact of each location in the network on the individual optimal portfolio composition, the model naturally advises consideration at multiple structural measures in order to rank players in terms of their risk-exposure.

The rest of the paper is organized as follows. In Section 2, we introduce some of the notation used throughout the chapter. In Section 3, we describe the model and present two examples of risk-sharing network structure. In Section 4, we compare the equilibrium risk-exposures with those maximising a social welfare function. In Section 5, we discuss the implications of the model. Section 6 concludes.

2 The network setup

We consider a finite set of players $N = \{1, ..., n\}$, connected in a network $G(N, L)$ with L the set of links or edges, describing their reciprocal relationships.[9] The players' linkages can be also represented in matrix notation through an adjacency matrix $A_{n\times n} = [l_{ij}]$, where $l_{ij} = 1$ whenever the nodes i and j are linked by an undirected edge while $l_{ij} = 0$ otherwise. We also assume that $G_{ii} = 0$ or no *loops* occur. The *degree* of a node $i \in G$, $d_i = \sum G_{ij} \forall j \in N$, is the number of links, or direct partners, the node i owns. For simplicity we will consider only connected components, i.e. $G_{ij} \neq 0$ for at least one $j \in N$. We define with *path* a non-empty graph $P = (V, E)$ of the form $V = \{1, ..., n\}$ and $E = \{l_{1,2}, ..., l_{n-1,n}\}$, where with $V \subseteq N$ we define a subset of nodes in the network $G(N, L)$ and with $l_{i,j}$ the specific linkage or edge between the nodes i and j belonging to the subset V. To define a "structural symmetry", or

[9] With g we generally define a connected structure or component while with G we refer to a generic network, not necessarily connected.

the case of two or more nodes symmetrically located in a graph, we use the notion of graph *automorphism*. Formally an automorphism is a one-to-one mapping, τ, from N to N of a graph $g(N, L)$ such that $< i , j > \in L$ if and only if $<\tau(i) , \tau(j) > \in L$. We can define an automorphism equivalence between i and j , $i \equiv^{AE} j$, whenever there exists some mapping τ such that $\tau(i) = j$, and the mapping τ is an automorphism. Thus, if we find such automorphism and we are interested in studying the impact of the link-structure on the agents, we can simplify the analysis and select specific players of the graph representing distinct equivalence classes. Throughout the chapter we will use the concept of network *density* to differentiate between different network structures. The density of a connected graph $G(N, L)$ of $|N| = n$ nodes is given by $2|L|/n(n-1) \in [0, 1]$, with $(n-1)$ being the minimum number of linkages in a component to be connected, and $n(n-1)/2$ the maximal one. Hereafter we define some specific graph of order n as following: the *complete graph*, or a graph where the nodes are all connected to each other, is indicated with K_n, the *path graph*, or n nodes connected in a line, with P_n, and the *star graph*, or a central node connected to $n-1$ peripheral nodes, with $K_{1,n-1}$.

3 The model

Consider a finite set of homogeneously risk-averse players, $N = \{1, ..., n\}$. There are two periods, t and $t+1$. Each player i simultaneously decides at time t how to allocate a unit capital between a risky asset, share x_i, and a risk-free one, $(1 - x_i)$. In particular, the risk-free asset is assumed to have zero variance, while the risky $\sigma_R^2 > 0$. Investment in the risk-free asset at t yields exactly the same amount at $t+1$, while the risky asset has positive expected returns px_ih, with $p \in (0, 1)$ the probability of return $h > 1$ at $t+1$. The players maximize their expected profit by choosing the optimal portfolio composition at t. The risky assets are assumed to be identical in variance and in the expected return for all the players in N, with a correlation coefficient $\varphi \in [0, 1]$.

The players are connected by a network $G(N, L)$, describing the liquidity flow-paths between the peers composing it. Therefore, at $t+1$, the agents are assumed to transfer liquidity to each other in order to comply with a given *equal-sharing* rule. Informally, at $t+1$ the players' transfers equalize the income "post-transfer" of all the agents belonging to the same component. Finally, we assume that the network structure is common knowledge and the players cannot deviate from the risk-sharing rule.

3.1 Fixed network structure and $\varphi = 0$ case

We start by considering the case where there is no correlation between the risky-assets, $\varphi = 0$, and no friction on the liquidity flow between players, i.e. the network $G(N, L)$ is constant over time. For a generic player $i \in N$, the expected income Y computed at time t and without considering the transfers at $t + 1$ is

$$E[Y_{it}] = x_{it}ph + (1 - x_{it}) \tag{1}$$

with variance equal to $Var[Y_{it}] = \sigma_R^2 x_{it}^2$. The players' instantaneous preferences are described by the following mean-variance utility function

$$U_{it}(C_{it}) = E[C_{it}] - \frac{\alpha}{2} Var[C_{it}] \tag{2}$$

where $\alpha > 0$ is the coefficient of absolute risk-aversion and C_{it} is the expected consumption at time t.[10] Once we take into account the transfers to and from the rest of the peers at $t + 1$, for each i, the expected income post-transfer becomes

$$E_{it}[I_{it}] = \frac{x_{it}ph + (1 - x_{it})}{n} + \frac{1}{n} \sum_{j \neq i} [x_{jt}ph + (1 - x_{jt})] \tag{3}$$

with I_{it} the income post-transfer. On the other hand, the portfolio's variance for each i becomes

$$Var[I_{it}] = \frac{x_{it}^2 \sigma_R^2}{n^2} + \frac{\sigma_R^2}{n^2} \sum_{j \neq i} x_{jt}^2 \tag{4}$$

We assume that (2) is twice continuously differentiable, increasing, and strictly concave in $x \in [0, 1]$, and thus there exists a unique maximum. In particular, we obtain the following result.

Lemma 1. *Let $G(N, L)$ be a connected network structure with $|N| \geq 2$. The equilibrium profile implies unique and homogeneous individual risk-exposures given by*

$$x_{it}^* = min\{1,\ n\frac{(ph - 1)}{\alpha \sigma_R^2}\}$$

for all $i \in N$.

Proof. It comes directly as solution of the optimization problem.

[10] Before transfer we can assume $C_{it} = Y_{it}\ \forall i \in N$.

From now on, we simplify the notation and define $k \equiv \dfrac{(ph-1)}{\alpha\sigma_R^2}$.[11] This standard result states that the optimal capital-share invested in a risky asset is function of its marginal return, of its variance, of the agent's risk-aversion degree, and finally of the size of the risk-sharing community. All things being equal, players belonging to the same risk-sharing group are able to increase their optimal risk-exposure proportionally to the size of their community. Allowing the transfers between agents to reach any player of the network without any friction, the individual locations in the component do not differentiate the players' portfolio composition. In the next section we are going to relax this specific assumption.

3.2 Structural frictions: Probabilistic Network Structure

Assume now that the connected network structure $G(N, L)$ at time t is itself "volatile", or alternatively put, any existing linkage $l_{ij} \in L$ at time t would not be certain at $t+1$. Define with G^t the network structure G at time t.[12] Formally, we define

$$\delta \equiv Pr[l_{ij} \notin G^{t+1} \mid l_{ij} \in G^t] \in (0,1) \; \forall l_{ij} \in G^t$$

with each probability weight δ independent and identically distributed. It is intuitive that the agents should now discount the possibility that any existing liquidity channel connecting a pair of players could decay or disrupt for some exogenous probability, and modify their optimal risk-exposure accordingly.[13] However, two non-isomorphic locations in G may have different impacts on the choices of their relative players. To clarify with an example, suppose a *star* graph, $K_{1,n-1}$, of order $n \geq 3$. At t, the centre of the star is directly linked to $(n-1)$ partners and therefore, a single link is enough to receive (and send) a transfer from (and to) each peripheral node. Conversely, for each peripheral player, both the link with the central agent and those with the rest of $(n-2)$ peripheral players are essential to receive and send transfers with the rest of community, i.e. the single disruption of the link with the central player is able to prevent any transfer flow from or to any other peer in the network. Therefore, it is clear that the optimal portfolio compositions of these two players will differ and in particular, the centre of the star will optimally choose higher risk-exposure than any peripheral node.

[11]Hereafter we will only report the optimal solutions lower than 1.

[12]With abuse of notation hereafter we define G^t simply with G.

[13]This could be the case when the existing informal insurance schemes suffer from imperfect monitoring and/or there could be various periodically exogenous shocks forcing a player to not share liquidity with other peers.

3.2.1 The n_{it}^{-1} measure

By assuming probabilistic link-structures we raise the following question: with how many peers could a player expect to risk-share in the next period? Alternatively put, what is the expected size of the risk-sharing component which a player could expect in the following period given the present link-structure?[14]

We define the measure $n_{it}^{-1} \equiv E_{it}[\frac{1}{n} \mid G]$, or the number of peers with whom a player $i \in G$ is expecting to risk-share in the following period, given a specific network structure G at time t and a *decay* probability coefficient $\delta \in (0, 1)$ related to each linkage. It is intuitive to expect the measure as function, among other things, of the average distance between the players; if the transfer from a player i would need a high number of linkages to reach a partner j, the relative probability to reach the target player j would be relatively low. Moreover, it is also intuitive that the n_{it}^{-1} measure should monotonically decrease with respect to the density of the network; if a player's transfer can flow through more than one path to reach the rest of the peers, then the chances of reaching its targets would be relatively high.

Define with $\mathcal{G}_{n,\delta}$ the probability space $\{\Omega, P\}$ of all the possible spanning subgraphs of a complete graph K_n, and for any event $A \subseteq \Omega$, the probability is $P(A) = \sum_{w \in A} P(w)$. The probability of observing any graph $G \in \mathcal{G}_{n,\delta}$ is $P(G) = \delta^{N-l}(1-\delta)^l$, with $l \equiv |L|$. Assume now a generic graph $G_n \neq K_n$. In particular, define with $G_{n,\delta}$ our new probability space of all the possible spanning subgraph of G_n. For a generic network G_n of order $n \geq 3$, we could potentially observe more than one combination of paths containing i and $m-1$ other nodes, and thus the computation of the n_{ij}^{-1} score for relatively large and dense networks could be computationally hard. Define with $H_m \subseteq G_{n,\delta}$ any graph of order m which includes a specific node $i \in G$, and with $P(A)$ the probability of observing the event A consisting of all the H_m subgraphs. We label with abuse of notation $P(A) \equiv P(H_m)$. Thus, we can write

$$n_{it}^{-1} \equiv E_{it}\left[\frac{1}{n} \mid G\right] = \sum_{m=1}^{n} \frac{1}{m} P(H_m) \tag{5}$$

It is easy to see that at the limit, for $\delta \to 1$ and $\delta \to 0$, the measure would score 1 and $1/n$ respectively, neutralizing any potential structural impact of G.[15] The following theorem can simplify the computational problem of computing the probability of observing a subset H_m, counting the number of edges to *cut* to isolate a subset of nodes in any generic graph $G(N, L)$.

[14]See Figure 10 in Appendix for an example of all possible subgraphs containing a specific node.

[15]Intuitively if $\delta \to 1$ in the next period any player belonging to the network expects to be isolated and therefore not to risk-share liquidity with other peers. Conversely, if $\delta \to 0$, any player in the given network reasonably expects in the following period to risk-share with other player of the same structure. In both cases the network architecture, and the relative players' location, does not impact their optimal choices.

Define with $|\partial(X)|$ the edge cut isolating the subset $X \subseteq N$ and with $e(X) = e(X, X)$ the number of edges within the vertexes of a subset $X \subseteq N$.

Theorem 1. ***(Bondy and Murty [2008])*** *For any graph $G(N, L)$ and any subset X of N, $|\partial(X)| = \sum_{i \in X} d(i) - 2e(X)$.*

Proof. The total degree of a subset of vertexes $X \subseteq V$ is given by $d(X) = \sum_{v \in X} d(v) = e(X, Y) + 2e(X, X)$, with Y defining the subset of nodes $Y = V \backslash X$. The $e(X, X) = e(X)$ is multiplied by the constant 2 since each edge between the vertexes in X has two distinct ends (recall that we are assuming no loops in our setting). Thus, $d(X) - 2e(X) = e(X, Y)$ and being $\partial(X) = E[X, Y]$ or $|\partial(X)| = |E[X, Y]| = e(X, Y)$, then $\sum_{v \in X} d(v) - 2e(X) = |\partial(X)|$. □

For instance, $|\partial(X)|$ could help us compute the probability of observing one H_m, or $p(H_m, L') = \delta^{|\partial(X)|}(1 - \delta)^{e(X)}$. Intuitively, higher $|\partial(X)|$ means that the subset of nodes X has a lower chance of being disconnected from the rest of the peers $N \setminus X$, or in other words, higher subnetwork resilience for any δ. Finally, as previously stated, for any probability space, the number of subsets of order m, $|H_m|$, could be greater than 1. Therefore, we can write $P(H_m) = \sum_k p(H_m)$ with $k = 1, ...K$ defining one of the K subsets H_m.

Making use of Theorem 1 and of one of the results presented in Mohar [1992], we can highlight the first of the three properties proposed of the n_{it}^{-1} measure. Define with $D(G)$ the diagonal matrix with the vertex degrees $d(i)$ on the diagonal, and recall that $A(G)$ is the adjacency matrix describing the network G. The Laplacian matrix is then $L(G) = D(G) - A(G)$, and it is positive, semidefinite, and symmetric with smallest eigenvalue $\lambda_1 = 0$.[16] Therefore, given an edge-cut $|\partial(X)|$ and cardinalities $|X| = n'$ and $|N| = n$ with $n' < n$, the *edge-density* of $|\partial(X)|$ is defined as

$$\rho(X) = \frac{|\partial(X)|}{n'(n - n')}$$

In particular, we know that this score is bounded below and above by $\lambda_2(G)/n \leq \rho(X) \leq \lambda_n(G)/n$ (Mohar [1992]). Combining this result with Theorem 1, we can therefore find the boundaries for the link-density of any subset $X \subset N$ as follows

Property 1. *Let $G(N, L)$ be a connected graph of order $|N| = n$ and $X \subset N$ a subset of connected nodes. The edge density of X is bounded by*

[16] Using this notation λ_n is the maximal eigenvalue of $L(G)$. See Mohar and Alavi [1991] for more details.

$$\frac{1}{2}\left(\sum_{i\in X} d(i) - \lambda_2\tilde{n}\right) \geq e(X) \geq \frac{1}{2}\left(\sum_{i\in X} d(i) - \lambda_n\tilde{n}\right)$$

with $\tilde{n} \equiv n'(n-n')/n$

A direct consequence of this property is the following. Suppose we have a connected graph G with eigenvalues of its relative Laplacian matrix which are non-trivial and relatively close to each other, i.e. the difference $\lambda_n - \lambda_2$ is particularly small. Then we know that the boundaries for all edge cuts $|\partial(X)|$ of vertex sets X with same cardinality are approximately the same. In other words, for any given connected G, without computing the precise score n_{it}^{-1} for each i, we are able to say wether the scores n_{it}^{-1} for each $i \in N$ are close to each other, i.e. if the nodes expect to be connected to the same number of peers on average.

3.2.2 Example: 2-regular cycle graph

Due to its computation tractability, we report as an example the computation of the n_{it}^{-1} *scores for any player* $i \in G(N, L)$*, with* G *a 2-regular graph and* $|N| = n \geq 3$ *players, e.g. a cycle graph (see Figure 2.2). It is easy to see that the* $|\partial(X)| = 2$ *for any set* H_m*'s size* $m \in [2, n-1]$*, and thus* $p(H_{n-1}) = \delta^2(1-\delta)^{m-1}$*. For the same order* m*,* $P(H_m) = \sum p(H_m) = m \cdot p(H_m)$*, or* $P(H_m) = m[\delta^2(1-\delta)^{m-1}]$*. Finally, the biggest subset* X_m *with order* $m = n$ *can be connected in two ways by* $(n-1)$ *linkages, and in one way through the entire cycle. Summing up, for any cycle graph of order* n *the* n_{it}^{-1} *score can be computed as following.*

$$n_{it}^{-1} = \left[\sum_{m=1}^{n-1} \delta^2(1-\delta)^{m-1}\right] + \delta(1-\delta)^{n-1} + \frac{1}{n}(1-\delta)^n \qquad \forall i \in G$$

We remark that Karakashian et al. [2013] proposes an algorithm to compute all the k connected subgraphs of a fixed size m of a given graph G. Moreover, in line with the centrality measure proposed by Estrada and Rodríguez-Velázquez [2005], and in contrast to the most common measures such as the *degree*, the *closeness*, the *betweenness*, and the *eigenvalue* centrality, n_{it}^{-1} scores its highest value in a complete network.[17]

[17] The measure presented here is related to centrality measure proposed by Estrada and Rodríguez-Velázquez [2005], defined also as *subgraph centrality*. The authors discuss a measure which ranks the nodes of a given connected graph G by the number of possible walks in G which they could be part of, negatively discounted by their lengths. Given the adjacency matrix A, the number of closed walks of length k starting and ending on a vertex $i \in G$ is given by the *local spectral moment*, $\mu_k(i) = \left(A^k\right)_{ii}$, that is simply the ith row element of the

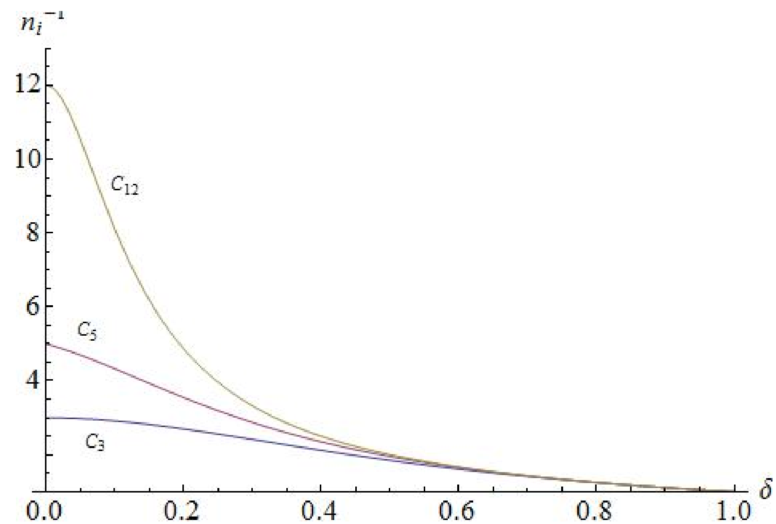
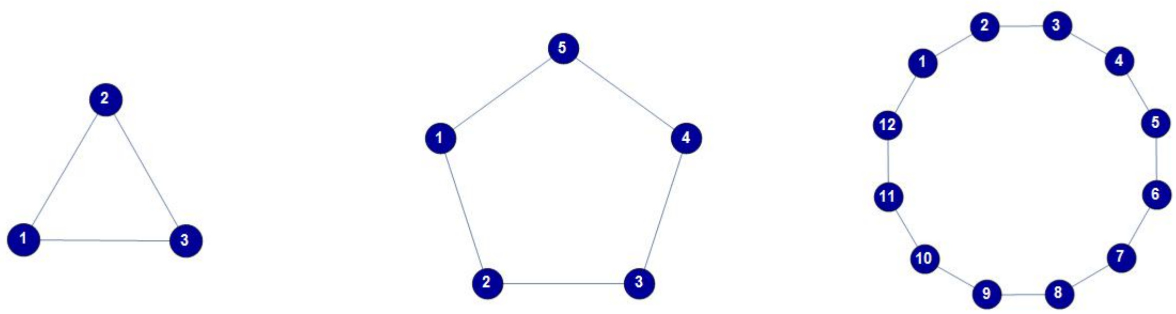


Figure 1: Expected cycle's size against δ. Differences among the cycle graphs of different order are observed only for relatively small δ coefficients.



(a) C_3 **(b)** C_5 **(c)** C_{12}

Figure 2: Cycles of order 3, 5, and 12.

Finally, we state the following two properties, direct consequences of the respective impacts of the graph density and of the geodesic distance between the nodes. In particular, let $G(N, L)$ be a connected graph, then for all $i \in G$.

Property 2. *The n_{it}^{-1} score monotonically decreases with respect to $|L|$, and monotonically increases with respect to the average distance $\bar{d}_i = \frac{1}{n(n-1)} \sum_{j \neq i} d_{ij}$ between i and any other player $j \in G$.*

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diagonal of the matrix A^k; the subgraph centrality score of a node $i \in G$ is then normalized as $SC(i) \equiv \sum_k^{\infty} \frac{\mu_k}{k!}$. However, while this measure could help to rank the nodes composing a given graph, the final n_{it}^{-1} score aims to give the exact expected size of the surviving subcomponent.

Property 3. *For any $\delta \in (0,1)$, $n_{it}^{-1} \in [1/\bar{n}, 1/\underline{n}]$, or the n_{it}^{-1} measure is bounded below by $1/\bar{n}$, the score of a node in a complete network K_n, and above by $1/\underline{n}$, the score of a peripheral node in a path graph P_n. Moreover, $\lim_{\delta \to 0} n_{it}^{-1} = 1/n$ and $\lim_{\delta \to 1} n_{it}^{-1} = 1$ for all $i \in G$.*

Intuitively, by increasing the number of links (Property 2), the number of possible channels guaranteeing liquidity flows among the players also rises. However, for each player, the marginal impact of a relatively distant connection between two different peers would be lower than a closer one. This implies that the structures which magnify both these features are the complete and path graphs (Property 3); the first guarantees the maximal number of linkages and thus the lowest distance between the peers, while the path graph is the connected structure with the lowest number of links and highest average distance between nodes.

3.3 The $\varphi > 0$ case and probabilistic structure

In this section we analyse the impact of a probabilistic network structure (flow frictions), allowing also for positive asset correlation levels summarized by the coefficient $\varphi \in (0,1]$. On the one hand, it is easy to see that the positive asset-correlation does not influence the post-transfer expected income. On the other hand, for each player i the portfolio's variance becomes

$$Var[I_{it}] = (n_{it}^{-1})^2 \sigma_R^2 \left(x_{it}^2 + \sum_{j \neq i} \theta_{ij,t} x_{jt}^2 \right) + 2\varphi \sigma_R^2 n_{it}^{-1} x_{it} \sum_{j \neq i} n_{jt}^{-1} \theta_{ij,t} x_{jt}$$

Solving the new individual optimization problem, we obtain a system of best reply functions for all $i \in N$

$$x_{it}^*(G) = \frac{1}{n_{it}^{-1}} \left(k - \varphi \sum_{j \neq i} n_{jt}^{-1} \theta_{ij,t} x_{jt}^*(G) \right) \tag{6}$$

We can rewrite this system in matrix notation, as $(kI_n + \varphi \bar{D})\mathbf{x} = \mathbf{1}$. I_n is a square diagonal matrix with n_{it} elements in the diagonal, $\bar{D} = ND$ where D is a symmetric and invertible matrix[18] composed by 0 elements in the diagonal and by probability weights $\theta_{ij,t}$ function of the distance between i and j otherwise[19]. Finally N is a square matrix with 0 elements in the diagonal and n_{jt}^{-1}/n_{it}^{-1} for each ijth element. The next result follows.

[18] Note that the matrix D is a *distance matrix*.

[19] In the case of tree graphs, $\theta_{ij,t} = (1-\delta)^{d_{ij}}$, while if there are multiple paths connecting a pair of nodes, we need to consider all the alternatives channels between them. The risk taken by one single player could in fact spread through all the existing channels.

Proposition 1. *An equilibrium profile exists, is unique, and implies individual risk-exposures given by* $\boldsymbol{x} = (kI_n + \varphi\bar{D})^{-1}\boldsymbol{1}$.

Proof. The matrix $(kI_n + \varphi\bar{D})$ is strictly diagonally dominant, hence non-singular and invertible. Therefore, a Nash equilibrium exists, is unique, and is defined by the vector $\mathbf{x} = (kI_n + \varphi\bar{D})^{-1}\mathbf{1}$. □

The main difference with the previous result is that now each player's optimal share of the capital invested in the risky asset is partially function of the risk taken by the rest of the players. Each individual best reply function 6 is separable with respect to the structural impact of the n_{it}^{-1} score and the "peer effect". In particular, the x^* takes into account for each i both the positive risk-pooling effect (first part of 6) and the negative impact due to the partially substitute risk taken by a $j \neq i$ player (second part multiplying φ). Alternatively said, the frictions on the transfers between nodes, summarized for each i by the n_{it}^{-1} score, decrease the risk-pooling effect but also the (negative) impact of correlated risk taken by other players.

It is easy to see that when $\delta \to 1$, then $1/n_{it}^{-1} \to n$, or when there are no network frictions each player expects to risk-share with other $n-1$ players. In such a case, $x_{it} \approx x_{jt}$, and thus the optimal risk exposure for each i becomes $x_{it}^* \approx nk/(1+\varphi(n-1))$.[20] Conversely, when $\delta \to 0$, then $1/n_{it}^{-1} \to 1$, or alternatively put, if each link $ij \in L$ existing at t decays with certainty at $t+1$, the network impact on the optimal individual choices will be null for each i, $x_{it}^* \to k$. In summary, for intermediate $\delta \in (0,1)$ values, the location of a player $i \in G$ affects her optimal choice x_{it}^* both through the n_{it}^{-1} score, and the optimal choices x_{jt}^* of her peers j (peer-effect).

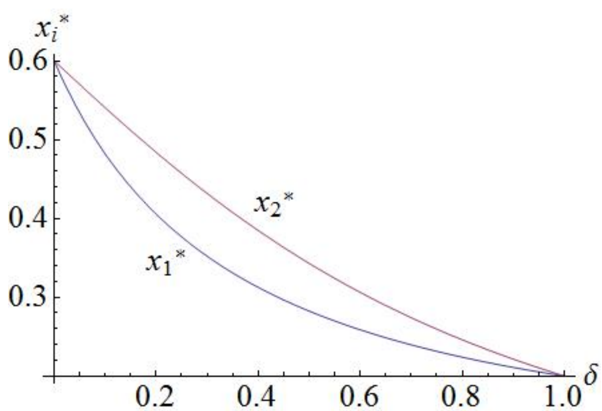
3.4 Examples

Suppose a path graph P_3 of order 3 such as the one in Figure 2.3. Player 2 connects players 1 and 3. We can just focus the analysis on choices of players 1 and 2, since 3 belongs to the same equivalence class of the node 1. Throughout, we will assume $k = 0.2$. We start by considering the case $\varphi = 0$. The optimal x_{it}^* (vertical axis) for each i as function of the parameter δ (horizontal axis) is depicted in Figure 4a. When $\delta \in (0,1)$, player 2 benefits more from risk-pooling than players 1 and 3, or in other words, player 2's location lets him invest a proportionally higher share of capital in the risky asset.

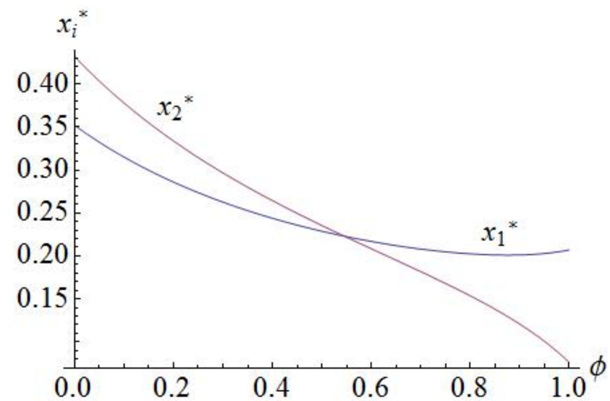
[20] In such cases, we need to assume $\varphi < 1$, such that the vectors composing $\mathbf{C}$ are linearly independent and therefore $|\mathbf{C}| \neq 0$, or $\mathbf{C}$ is invertible.

Figure 3: P_3 graph

Consider now the case $\varphi \in (0, 1]$ and assume $\delta = 0.3$.



(a) $\varphi = 0$



(b) $\varphi \in (0, 1)$ and $\delta = 0.3$.

Figure 4

The relative players' vector of expected component-size is then $\mathbf{n'} = (0.57, 0.46, 0.57)$. Player 2's advantage with respect to 1 and 3 persists only at levels of φ which are small enough; for relatively high asset correlation, being "closer" to the rest of the peers becomes detrimental since the positive effect due to the high $1/n_{2t}^{-1}$ score does not compensate the negative impact due to the risk taken by 1 and 3. These players are structurally less exposed to the risk of their peers and consequently they are able to invest more in the risky-asset.[21]

[21]We are not going to report the plots for different δ parameters since the results seem to change neither qualitatively nor substantially.

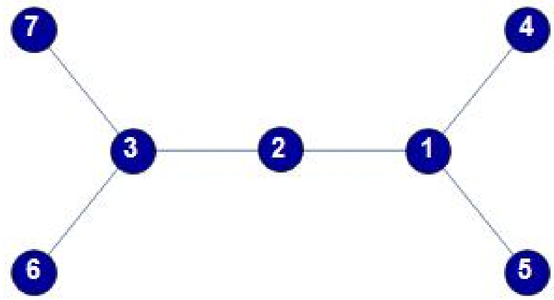
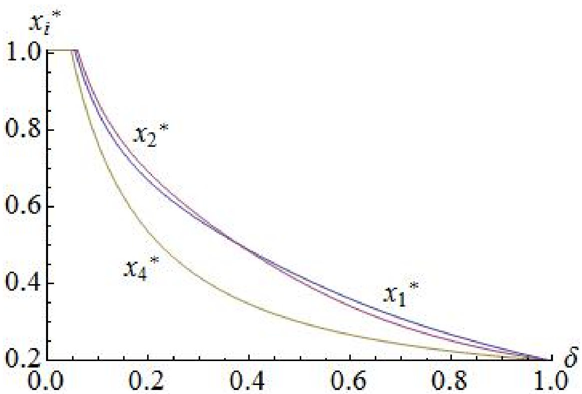
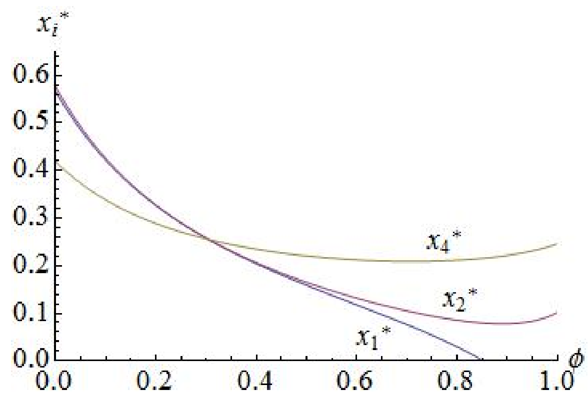


Figure 5: T_7 graph

Suppose now a tree graph, T_7 of order 7 as in Figure 2.5. Players 1 and 3 own the highest degree score, $d_{1,3} = 3$, while players $\{4, 5, 6\}$ the lowest, $d_{4,5,6,} = 1$. Player 2 owns a relatively intermediate degree score, $d_2 = 2$, but he is located in a *bridge* position between two automorphic subcomponents. This classification also describes our three equivalence classes composing the graph. Consider again the case $\varphi = 0$. The optimal x_{it}^* for intermediate levels $\delta \in (0, 1)$ are represented in Figure 2.6a.



(a) $\varphi = 0$



(b) $\varphi \in (0, 1)$ and $\delta = 0.3$.

Figure 6

Interestingly, for relatively low levels of δ, players 2, 1 and 3 invest similar capital share in risky-assets, with a slight difference in favor of 2. Increasing δ, players $\{1, 3\}$ own the lowest n_{it}^{-1} scores and, given $\varphi = 0$, this means that they will have the highest risk-exposure among the players. Consider now the case $\varphi = (0, 1]$. As previously, for each specific value of δ, there exists a level of positive risk-correlation after which the peripheral locations appear to be more beneficial than the central ones; symmetrically, the peripheral nodes $\{4,5,6,7\}$ become the players investing more in the risky alternative.

4 Efficient risk allocations

From the previous sections it is clear that the players do not internalize the cost of their individual optimal portfolio choices on the rest of the peers; assuming positive asset correlation φ, the risk exposure of one player is partially and reciprocally substitute for the risk taken by the rest of the players.

Consider the problem of a central planner who aims to maximize the following utilitarian social welfare function with respect to the vector of individual risk-exposures $\mathbf{x}$ and given a fixed connected network $G(N,L)$:

$$\max_{\mathbf{x}} W(G) = \sum_{i \in G} U_{it} = \sum_{i \in G} \left[E_{it}(I_{it}) - \frac{\alpha}{2} Var(I_{it}) \right] \tag{7}$$

We define as *efficient* an allocation vector $\mathbf{x^W}$ which maximizes $W(G)$. It follows the next result.

Proposition 2. *Let $G(N,L)$ a connected probabilistic network structure and $\varphi \in (0,1]$ the positive correlation parameter. The equilibrium allocation $\mathbf{x}^*$ is not necessarily efficient.*

In particular, the final sign of the difference $(x_{it}^{W*} - x_{it}^*)$ $\forall i \in G$ will depend on player i's location and on the parameters φ, δ, and α. For intermediate $\delta \in (0,1)$, and α and φ high enough, $x_{it}^W < x_{it}^*$, suggesting that the "social cost" of each player's optimal risk-exposure would be relatively high. In other words, the combination of structural frictions on the transfer flows, high risk-aversion, and high correlation, leads to a general individual over-exposure to risky assets compared to the socially optimal level. On the other hand, keeping constant α and φ, and for low enough decay probability δ, the individual optimal risk-exposures $x_{it}^W \approx x_{it}^*$ due to the higher capacity of risk-absorption of the network. Finally, when the risk premium is large, the absorption capacity of G is high (high lowest eigenvalue λ_0), and the risk-aversion is low, eventually we obtain $x_{it}^W > x_{it}^*$, i.e. high risk premium, high risk-pooling effect, and low risk-aversion degree, all offset the negative externalities due to the peer effect and therefore in equilibrium we would observe under-exposure to risky-assets.[22]

[22]This last case is remarkably close to the result in Proposition 2 of Belhaj and Deroïan [2012]. The authors

5 Discussion

In the previous analysis, for positive correlation coefficients φ, we interpreted a linkage between two players as a channel of both liquidity and risk. It is intuitive to see that this double feature could be enough to prevent the use of a single structural measure in order to rank the nodes' location in the network in terms of risk as perceived by the respective players. All the standard centrality measures adopted in the literature, such as *degree* centrality, *closeness* centrality, *betweenness* centrality, and the family of *eigenvector* centralities (Borgatti and Everett [2006], Borgatti [2006], Bonacich [1987]) would not be able to rank the locations of players as function of the risk they face for varying correlation levels.

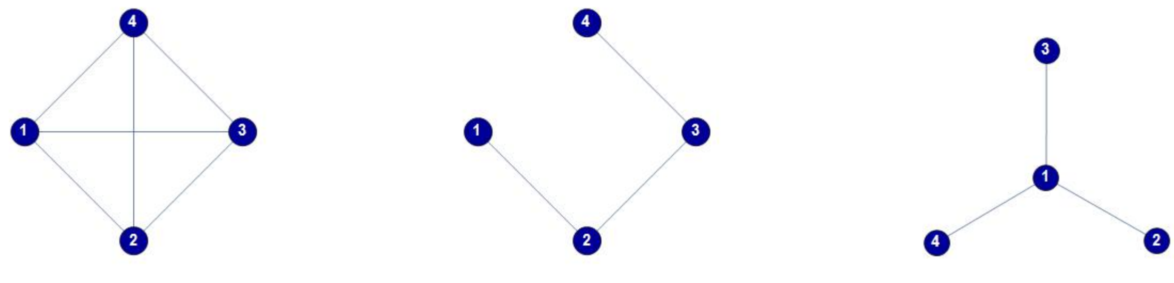
An interesting and useful network property has been recently highlighted by Bramoullé et al. [2014]. The authors show how the size, in absolute value, of the lowest eigenvalue associated to a graph G's adjacency matrix A, $|\lambda_0|$, gives information about specific structural features of G.[23] The graph of cardinality n with the smallest lowest λ_0 is the one with the highest number of links and lowest *triadic closure*, and the complete bipartite graph (*CB*), with partitions of nodes as equally sized as possible, particularly fits this profile. The authors also noted that a structure with a high number of triadic closures relatively decreases the peer effect, or the impact of the actions of the partners, thereby absorbing the shocks. Consequently, the complete graph, with the highest $\lambda_0 = -1$ and thus highest number of triadic closures, is the graph which minimizes the impact of the peers' actions. In the previous sections we remarked that, assuming a probabilistic structure, the complete graph K_n of order n is also the network structure which maximizes the n_{it} scores for all $i \in K_n$. Therefore, combining these results with Property 3 of the n_{it}^{-1} score, we deduce that the network structure which would lead to the highest aggregate risk exposure of the players will be the graph with the highest lowest eigenvalue (see Figure 7-8).[24]

We remark that this result shares one of the conclusions reached by Allen and Gale [2000], even if for a different model setting. In their financial contagion model the authors conclude that the complete network optimizes the risk-pooling effect, both diversifying the players' portfolio volatility and absorbing the negative liquidity shocks started by one of the nodes in

pointed out that this could describe the lack of investment in risky innovations observed in many developing villages (see Valente [2005]).

[23]The results presented by these authors use the Rayleigh-Ritz theorem and results by Doob and Cvetković [1979].

[24]An interesting structural feature owned by the star graph should be mentioned. From this specific network is particularly "easy" to create triadic closures: any marginal linkage activated between the nodes creates a new triadic closure. In particular, for n even, we can create enough triadic closures to totally neutralize the peer effect in a network, activating $(n-1)/2+1$ new edges ($(n-1)/2$ for n odd).



(a) K_4 **(b)** P_4 **(c)** $K_{1,3}$

Figure 7

the network. Complementary to this, our model implies that:

- The **complete graph** K_n maximizes n_{it} $\forall i \in N$ and minimizes the peer effect.
- The **path graph** P_n minimizes n_{it} $\forall i \in N$ and implies a relatively strong peer effect, though not maximal.
- The **star graph** $K_{1,n-1}$ is the tree graph which maximizes both n_{it} $\forall i \in N$ and the peer effect.

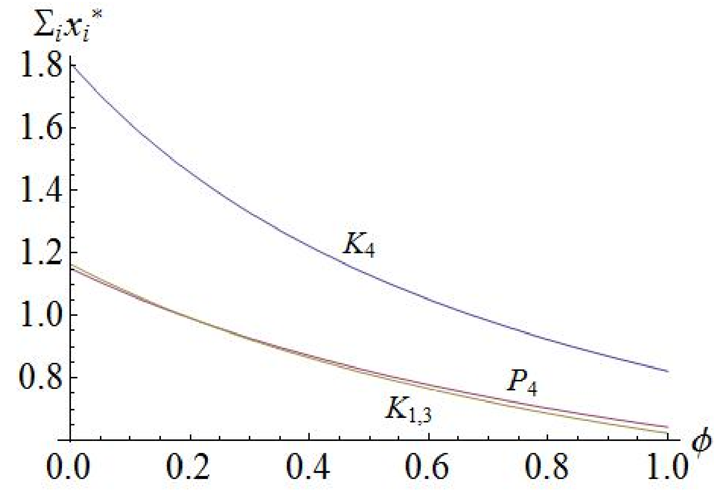


Figure 8: Total risk exposure in K_4, P_4, and $K_{1,3}$ graphs ($\delta = 0.6$). The total exposures in P_4 and $K_{1,3}$ are very close to each other due to the relative small order of the graphs.

6 Conclusions

We have discussed a simple model showing how specific frictions at link-level could differentiate the optimal risk-exposure of agents belonging to a risk-sharing community. All things being equal, players which are homogeneous in risk-aversion but located in non-isomorphic locations in the network may face different risk-pooling opportunities, and will therefore optimally choose heterogeneous risk-exposure levels. We investigated the impact of a positive homogeneous degree of asset correlation and highlighted that locations which benefit more from risk-pooling when the correlation level is low also become more exposed to systemic risk once we assume positive and high enough correlation levels. We argue that this non-monotonicity could challenge the efficacy of using of a single network centrality measure to rank players in terms of their risk-exposure. Furthermore, assuming structural frictions and a positive degree of asset correlation, we argue that complete networks are the graphs maximizing both the risk-pooling level of the community and the systemic risk absorption, and thus they might lead to the highest aggregate risk-exposure level. Finally, the players fail to internalise the cost of their risk-exposure level on the rest of their peers. As a consequence, the decentralised equilibrium profile might not necessarily be efficient.

Further extensions could refine the probabilistic setting proposed. A model extension could endogenize the link-decay probability δ, making it function of the specific risk exposure of the pair of players involved. Finally, it might be interesting to implement a strategic link-formation process; incentives to form new connections and the choice of the link recipients may be linked, among other things, to the respective asset correlation and thus different architectures may arise as function of this parameter.

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A Appendix

Proof of Property 2. The first part comes directly from the fact that $p_m(H_m)$ increases with the number of combinations of m nodes including i. Therefore, increasing $|L|$ the number of combinations strictly rises for any node in the graph. For the second part, we know by definition that a node j belongs to a spanning subgraph which includes the node i if and only if there exists a path between i and j. Assuming i.i.d. probability that each link decays at $t+1$, for any spanning subgraph $G' \subseteq G$ such that $i, j \in G'$ the probability to reach j from i computed at t is $(1-\delta)^{d_{ij}}$. Therefore, if d_{ij} increases, the same probability decreases, or alternatively put, the probability that j will belong to a spanning subgraph including i decreases, and thus all things being equal n_{it}^{-1} must rise. □

Proof of Property 3. Suppose a constant $\delta = \bar{\delta} \in (0,1)$ and G_n. By Property 1 and 2, n_{it}^{-1} score decreases with $|L|$ and increases with $\bar{d}_i$. The connected graph G_n with highest $|L|$ and shortest $\bar{d}_i$ for each $i \in G_n$ is K_n, or the complete network of order n. Therefore, $1/\bar{n}$ is the lower bound of n_{it}^{-1} for each i. On the other hand, the graph of order n with highest $\bar{d}_i$ and minimal $|L|$ is the path graph P_n. Thus, $1/\underline{n}$ is the upper bound of n_{it}^{-1}. Finally, for any G_n, it is straightforward to see that for $\delta \to 0$ then $n_{it}^{-1} \to 1/n$ since all the links to reach the n nodes survive at $t+1$ and thus there are no frictions in G. For $\delta \to 1$ no link survives in the following period and then $n_{it}^{-1} \to 1$ since each node i will be isolated. □

Proof of Proposition 2. Maximizing the (7) with respect to each individual risk-exposure leads to first order conditions for each $i \in G$ which we can decompose in two parts as follows,

$$\frac{\partial W(g)}{\partial x_{it}} = \underbrace{\frac{\partial E_{it}(I_{it})}{\partial x_{it}} - \alpha \frac{\partial Var_{it}(I_{it})}{\partial x_{it}}}_{A} + \underbrace{\frac{\partial E_{it}(I_{it})}{\partial x_{jt}} \frac{\partial x_{jt}}{\partial x_{it}} - \alpha \frac{\partial Var_{it}(I_{it})}{\partial x_{jt}} \frac{\partial x_{jt}}{\partial x_{it}}}_{B} = 0$$

A is the common part with the first order conditions individually solved by i, and B is the impact of x_{it} on j players' choices, internalized by the central planner. In particular,

$$B = \underbrace{\frac{1}{n_{it}} \sum_{j \neq i} \theta_{ij} \frac{\partial x_{jt}}{\partial x_{it}}}_{C} \underbrace{\left[ph - 1 - 2\alpha \left(\frac{\sigma_R^2}{n_{it}} \sum_{j \neq i} x_{jt} + \varphi x_{it} \sum_{j \neq i} \frac{1}{n_{jt}} \right) \right]}_{D}$$

Since $C < 0$, $B \neq 0 \Leftrightarrow D \neq 0$, or

$$2\alpha \left(\frac{\sigma_R^2}{n_{it}} \sum_{j \neq i} x_{jt} + \varphi x_{it} \sum_{j \neq i} \frac{1}{n_{jt}} \right) \neq ph - 1 \tag{8}$$

Therefore, if (8) holds, $B \neq 0$ and thus $x_{it}^W \neq x_{it}^*$. □

Corollary 1. *For any pair of players $(i, j) \in G$ such that $n_{it} > n_{jt}$, under certain conditions there exists a unique positive level φ_{ij}^* such that $x_{it}^* = x_{jt}^*$, and*

- $\varphi < \varphi_{ij}^* \Leftrightarrow x_{it}^* > x_{jt}^*$
- $\varphi > \varphi_{ij}^* \Leftrightarrow x_{it}^* < x_{jt}^*$

Proof. Suppose a graph G of order $n > 2$ and the relative system of b.r.f. 6. For each player i we have $x_{it} = n_{it}k - n_{it}\varphi \sum_{j \neq i} \frac{\theta_{ij}}{n_{jt}} x_{jt}$. Consider a pair of non-isomorphic players $i, j \in G$ such that $n_{it} > n_{jt}$. We can disentangle the total peer effect observed by any i by separating the risk received from j and the risk received from any other $j' \neq j, i$. Thus,

$$\begin{cases} x_{it} = n_{it}k - n_{it}\varphi \dfrac{\theta_{ij}}{n_{jt}} x_{jt} - n_{it}\varphi \displaystyle\sum_{j' \neq i,j} \dfrac{\theta_{ij'}}{n_{j't}} x_{j't} \\ x_{jt} = n_{jt}k - n_{jt}\varphi \dfrac{\theta_{ij}}{n_{it}} x_{it} - n_{jt}\varphi \displaystyle\sum_{j' \neq i,j} \dfrac{\theta_{jj'}}{n_{j't}} x_{j't} \end{cases}$$

We want to find the conditions for the existence of a unique level $\varphi = \varphi_{ij}^*$ for each pair (i, j) such that $x_{it}^* = x_{jt}^* = \bar{x}$. Rearranging the i and j's relative b.r.f. we obtain,

$$\bar{x} = n_{it} \frac{k - \varphi \displaystyle\sum_{j' \neq i,j} \frac{\theta_{ij'}}{n_{j't}} x_{j't}}{1 + n_{it}\varphi \dfrac{\theta_{ij}}{n_{jt}}} = n_{jt} \frac{k - \varphi \displaystyle\sum_{j' \neq i,j} \frac{\theta_{jj'}}{n_{j't}} x_{j't}}{1 + n_{jt}\varphi \dfrac{\theta_{ij}}{n_{it}}}$$

Label for simplicity $n_{it}k \equiv \alpha_i$, $n_{it} \sum_{j' \neq i,j} \frac{\theta_{ij'}}{n_{j't}} x_{j't} \equiv \beta_i$, and $n_{it} \frac{\theta_{ij}}{n_{jt}} \equiv \omega_i$ for the player i and similarly for j, $n_{jt}k \equiv \alpha_j$, $n_{jt} \sum_{j' \neq i,j} \frac{\theta_{jj'}}{n_{j't}} x_{j't} \equiv \beta_j$, and $n_{jt} \frac{\theta_{ij}}{n_{it}} \equiv \omega_j$. Then, we can rewrite the equality as $\frac{\alpha_i - \varphi\beta_i}{1 + \varphi\omega_i} = \frac{\alpha_j - \varphi\beta_j}{1 + \varphi\omega_j}$. Solving for φ, we obtain the quadratic expression $\varphi^2 A +$

$\varphi B + C$, where

$$\begin{aligned} A &\equiv \beta_i w_j - \beta_j w_i = \sum_{j' \neq i,j} \frac{x_{j't}}{n_{j't}} \theta_{ij}(n_{jt}\theta_{ij'} - n_{it}\theta_{jj'}) \\ B &\equiv \beta_i - \beta_j + \alpha_j \omega_i - \alpha_i \omega_j = \sum_{j' \neq i,j} \frac{x_{j't}}{n_{j't}} (n_{it}\theta_{ij'} - n_{jt}\theta_{jj'}) + k\theta_{ij}(n_{it} - n_{jt}) \\ C &\equiv \alpha_j - \alpha_i = k(n_{jt} - n_{it}) \end{aligned}$$

Note that the discriminant is

$$B^2 - 4AC = \left[\sum_{j' \neq i,j} \frac{x_{j't}}{n_{j't}} (n_{it}\theta_{ij'} - n_{jt}\theta_{jj'}) - k\theta_{ij}(n_{jt} - n_{it}) \right]^2 - 4k\theta_{ij}(n_{jt} - n_{it}) \sum_{j' \neq i,j} \frac{x_{j't}}{n_{j't}} (n_{jt}\theta_{ij'} - n_{it}\theta_{jj'})$$

and it is greater than 0 if either $A > 0$ and $\frac{n_{jt}}{n_{it}} \geq \sum_{j' \neq i,j} \frac{\theta_{jj'}}{\theta_{ij'}}$, or $4AC = B^2$ if $A < 0$, being C always negative for $n_{it} > n_{jt}$. Define the solution to the quadratic expression φ^*_{ij}. We can always exclude $\varphi^*_{ij} < 0$ having assumed φ positive so φ^*_{ij} is unique. Moreover, if $\varphi^*_{ij} = 0$, then $x^*_{it} > x^*_{jt}$, since we have started assuming $n_{it} > n_{jt}$, thus $\varphi^*_{ij} > 0$. □

Under certain conditions, for each pair of players (i, j) not isomorphically located in G, there exists a unique level of positive correlation $\varphi = \varphi_{ij}$ such that $x^*_{it} = x^*_{jt} = x^*$. However, this result is valid at *pair-wise* level, i.e. φ_{ij} is not necessarily equal for any pair $(i, j) \in G$. Therefore, the Corollary implicitly suggests that for different parameters φ, a *safer* location could become *riskier*, or more "exposed" to correlated risk, with respect to an alternative location. In other words, each player i's location in G characterizes her optimal risk-exposure, and the relative ratio x^*_{it}/x^*_{jt} for any pair $(i, j) \in G$ is not necessarily monotonically greater or lower than 1 for any value of $\varphi \in (0, 1)$.

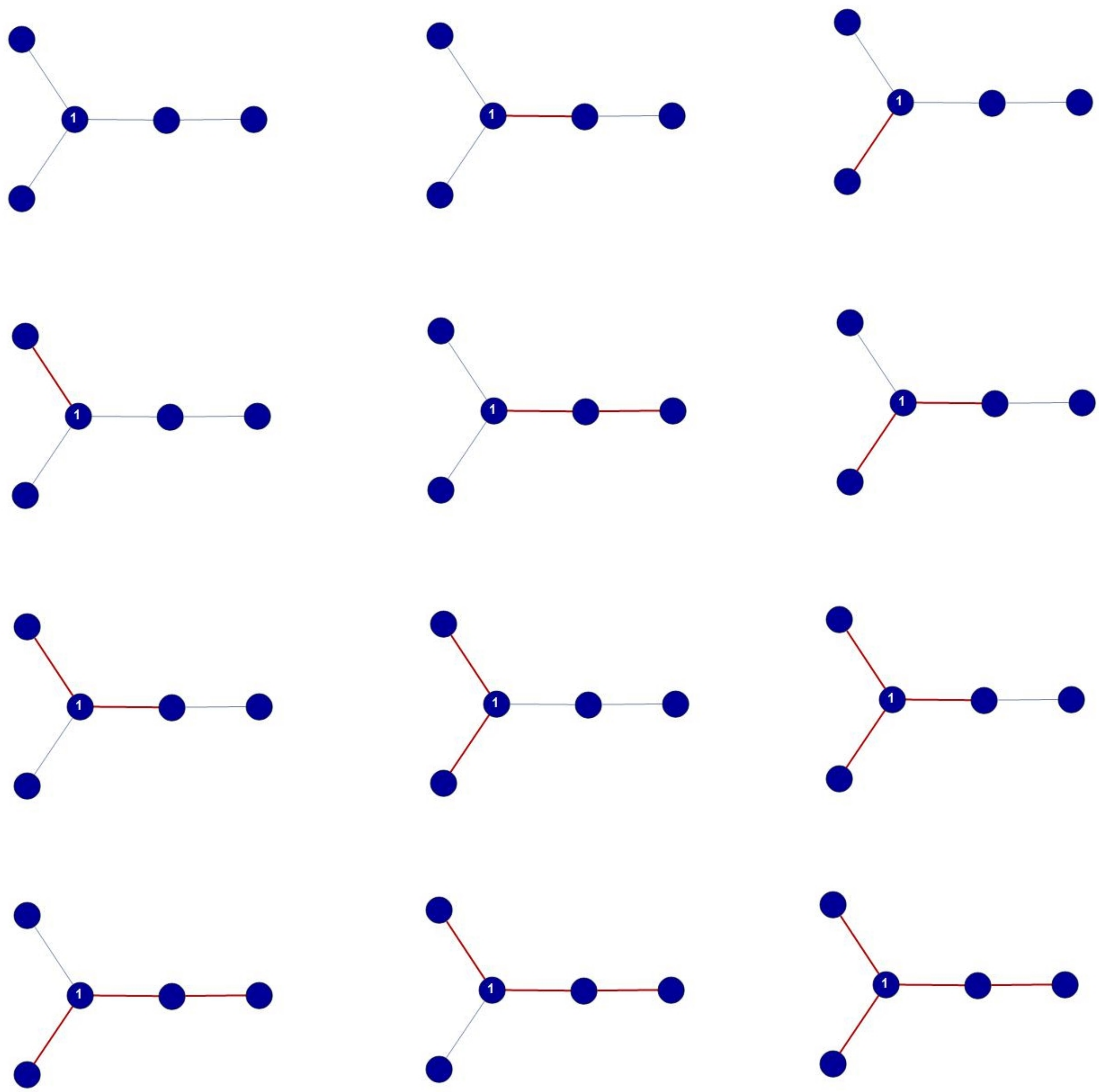

Figure 9: All possible subgraphs of different sizes containing node 1.