Bargaining Game

Seminar XII

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Bargaining Game: Nash Axiomatic Model

- Define with X the set of possible agreements and with D the disagreement outcome.

  As example, we could have
  $X = \{(x_1, x_2) | x_1 + x_2 = 1, x_i \geq 0\}$, $D = (0, 0)$. In words, suppose we are splitting a pie of size 1. If we do not agree on the splitting, we get 0.
- Each player i has preferences represented by u_i over $X \cup \{D\}$. The set of payoffs is U. Formally,
  $U = \{(v_1, v_2) | u_1(x) = v_1, u_2(x) = v_2 \text{ for some } x \in X\}$ and
  $d = (u_1(D), u_2(D))$.

Bargaining Game: Nash Axiomatic Model

- A **bargaining problem** is a pair (U, d) where $U \in \mathbb{R}^2$ and $d \in U$. We assume that U is convex and compact set, and that there exists some $v \in U$ such that $v > d$.
- The set of all possible bargaining problems is B.
- A **bargaining solution** is a function $f : B \longrightarrow U$.
- We will study bargaining solutions $f(\cdot)$ that satisfy a list of reasonable axioms.

Axioms

- **Pareto Efficiency:** A bargaining solution $f(U, d)$ is Pareto Efficient if there is no $(v_1, v_2) \in U$ such that $v \geq f(U, d)$ and $v_i > f_i(U, d)$ for some i.
- **Symmetry:** Let (U, d) be such that $(v_1, v_2) \in U$ if and only if $(v_2, v_1) \in U$ and $d_1 = d_2$. Then $f_1(U, d) = f_2(U, d)$. In words, if the players are identical, the agreement should not discriminate between them.

Axioms

- **Linear invariance:** Given a bargaining problem (U, d), consider a different one (U', d') for some $\alpha > 0, \beta$:

$$U' = \{(\alpha_1 v_1 + \beta_1, \alpha_2 v_2 + \beta_2) | (v_1, v_2) \in U\}$$

$$d' = (\alpha_1 d_1 + \beta_1, \alpha_2 d_2 + \beta_2)$$

Then, $f_i(U', d') = \alpha_i f_i(U, d) + \beta_i$. In words, any linear transformation of the payoffs should not change the outcome of the bargaining process.

Axioms

- **Independence of Irrelevant Alternatives:** Let (U, d) and (U', d) be two bargaining problems such that $U' \subseteq U$. If $f(U, d) \in U'$, then $f(U', d) = f(U, d)$.

Nash Bargaining Solution

Definition

We say that a pair of payoffs (v_1^*, v_2^*) is a Nash bargaining solution if it solves the following optimization problem:

$$\max_{v_1, v_2} (v_1 - d_1)(v_2 - d_2)$$

subject to $(v_1, v_2) \in U$ and $(v_1, v_2) \geq (d_1, d_2)$.

The solution exists, it is unique, and satisfies the axioms!

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- If we denote these shares by θ_i and θ_j then $\theta_i + \theta_j \leq 1$ is required for a meaningful solution of the game where each get $\theta_i \geq 0$ and $\theta_j \geq 0$ payoff. When $\theta_i + \theta_j > 1$ then $\theta_i = 0$ and $\theta_j = 0$.

Nash Product in Bargaining Game

$$\max U = (\theta_i - 0)(\theta_j - 0) \tag{1}$$

subject to

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$$L(\theta_i, \theta_j, \lambda) = (\theta_i - 0)(\theta_j - 0) + \lambda[1 - \theta_i - \theta_j]$$

First Order Conditions

First order conditions of this maximization problem are

$$\frac{\partial L(\theta_i, \theta_j, \lambda)}{\partial \theta_i} = \theta_j - \lambda = 0$$

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From the first two first order conditions $\theta_j = \theta_i$ and putting this into the third first order condition $\theta_i = \theta_j = 1/2$. Thus Nash solution of this problem is to divide the pie symmetrically into two equal parts.

Example of Bargaining Game

Suppose there is 1000 in the table to be split between two players. What is the optimal solution from a symmetric bargaining game if the threat point is given by $d(0,0)$? Using a Lagrangian function for constrained optimisation

$$L(u_1, u_2, \lambda) = u_1 u_2 + \lambda[1000 - u_1 - u_2]$$

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Thus, we obtain $u_2 = u_1 = 500$.

Example of Bargaining Game

The Nash bargaining solution is the values of u_1 and u_2 that maximise the value of the Nash product $u_1 u_2$ subject to the resource allocation constraint,$u_1 + u_2 = 1000$. This bargaining solution fulfils four different properties: 1) symmetry 2) efficiency 3) linear invariance 4) independence of irrelevant alternatives (IIA).

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- IIA implies irrelevant alternatives are not discussed in the game.

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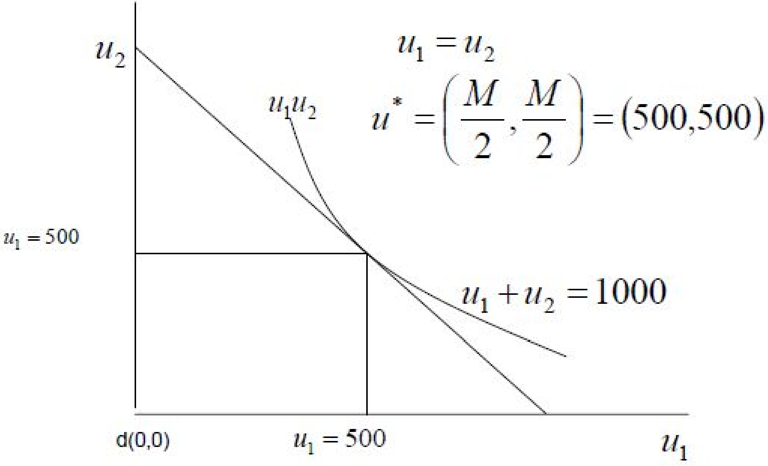

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$$L(u_1, u_2, \lambda) = (u_1 - d_1)(u_2 - d_2) + \lambda[1 - u_1 - u_2]$$

First order conditions of this maximisation problem are

$$\frac{\partial L(u_1, u_2, \lambda)}{\partial \theta_i} = u_2 - d_2 - \lambda = 0$$

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Thus, we obtain $u_1 = \frac{1}{2} + \frac{d_1 - d_2}{2}$ and $u_2 = \frac{1}{2} + \frac{d_2 - d_1}{2}$.

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Thus, we obtain $u_1 = \dfrac{1}{2} + \dfrac{d_1 - d_2}{2}$ and $u_2 = \dfrac{1}{2} + \dfrac{d_2 - d_1}{2}$.

If $d_1 = d_2$, then $u_1 = u_2 = 1/2$.

If $d_1 > d_2$, then $u_1 = 1/2 + K$ and $u_2 = 1/2 - K$ with $K = \dfrac{d_1 - d_2}{2}$.

Example

Suppose $u_1 + u_2 = 50.000$ and $d_1 = 15.000$ and $d_2 = 0$.

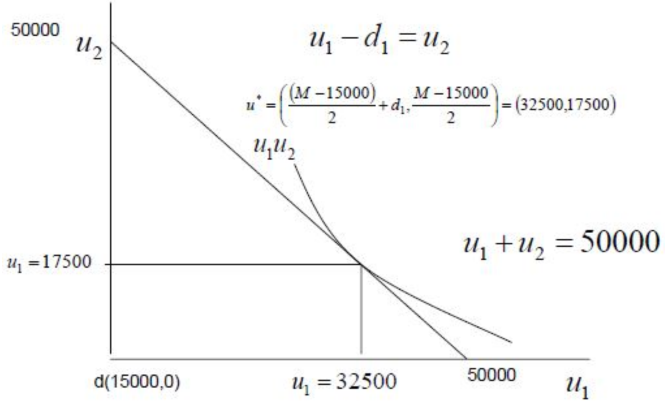

Figure: Bargaining Solution

Risk and Bargaining

A risk averse person loses in bargaining but the risk neutral person gains. Suppose the utility functions of risk averse person is given by $u_2 = (m_2)^{0.5}$ but the risk neutral person has a linear utility $u_1 = m_1$. $m_1 + m_2 = M$ and $u_1 + u_2^2 = 100$.

$$L(u_1, u_2, \lambda) = u_1 u_2 + \lambda[100 - u_1 - u_2^2]$$

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$$\frac{\partial L(u_1, u_2, \lambda)}{\partial \theta_i} = u_2 - \lambda = 0$$

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Thus, we obtain $u_2 = 5.77$, $u_1 = 66.6$, or $\theta_2 = 33.29$ and $\theta_1 = 66.6$.

Risk and Bargaining

Morale: do not reveal anyone if you are risk averse, otherwise you will lose in the bargaining!

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- **Efficiency:** two rational bargainers will not agree on a utility outcome $u = (u_1, u_2)$ if in the feasible set F there is another utility outcome $u' = (u_1', u_2')$ yielding higher payoffs for both of them.
- **Symmetry:** a symmetric game will have a symmetric agreement point $u = (u_1, u_2)$ with $u_1 = u_2$. (In a symmetric game the two players have exactly the same strategic possibilities and have exactly the same bargaining power. Therefore, neither player will have any reason to accept an agreement yielding him a lower payoff than his opponent's.)

General Properties of the Nash Bargaining Equilibrium

- **Linear Invariance:** because von Neumann-Morgenstern utility functions are behaviorally equivalent if one can be obtained from the other by an order-preserving linear transformation, if one game G' can be obtained from game G by subjecting player 1's utilities to an order-preserving linear transformation while keeping the other player's utilities unchanged, then a change in the utility outcome will not affect the physical outcome chosen.

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- **Independence of irrelevant alternatives:** let G be a bargaining game with conflict point, with feasible set F, and with agreement point u; and let G' be a game obtained from G by restricting the feasible set to a smaller set F' contained in F in such a way that d and u remain within the new feasible set F', d remaining the conflict point also for G'. Then u will be the agreement point also for this new game G'.