Microeconomic Analysis

Seminar 2

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Outline

PART I: Optimization

- Existence of an optimum
- Necessary conditions for an interior optimum (One-Many variables)
- Local optimum (One-Many variables)
- Global optimum (One-Many variables)

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PART II: Optimization-Equality constraint

- Two variables - one constraint.
  1. Necessary conditions for an optimum
  2. The Lagrange multiplier
  3. Sufficient conditions for an optimum
  4. Conditions under which a stationary point is a global optimum

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We can write the decision maker (DM) problem as,

$$\max_a u(a) \text{ subject to } a \in S$$

where $u(a)$ is the DM's payoff function (utility) over her action (a) and S is the set of feasible actions.

More generally we can write the optimization problem as

$$\max_{\mathbf{x}} f(\mathbf{x}) \text{ subject to } \mathbf{x} \in S$$

where $\mathbf{x}$ is a n-vector $(x_1, ..., x_n)$ and S is the set of these n-vectors. We call f the **objective function**, x the **choice variable**, and S the **constraint set** or **opportunity set.** We define the variable $\mathbf{x}^*$ the **maximizer** iff $f(\mathbf{x}) \leq f(\mathbf{x}^*)\, \forall \mathbf{x} \in S$.

Definition

The variable $\mathbf{x}^*$ is a local maximizer of the function f subject to the constraint $\mathbf{x} \in S$ if there is a number $\epsilon > 0$ such that $f(\mathbf{x}) \leq f(\mathbf{x}^*)\ \forall \mathbf{x} \in S$ for which the distance between $\mathbf{x}$ and $\mathbf{x}^*$ is at most ϵ.

Recall that any maximization problem can be transformed in a minimization problem transforming the objective function $f(\mathbf{x})$ in $-f(\mathbf{x})$.

Consider again the problem

$$\max_{\mathbf{x}} f(\mathbf{x}) \text{ subject to } \mathbf{x} \in S$$

Before computing the solution we could ask whether the problem has a solution.

Existence of an optimum (cont.)

The following are cases in which there is **NO** solution to the problem,

- $f(x) = x$ and $S = [0, \infty)$ (S is the set of all nonnegative real numbers).
- $f(x) = 1 - \frac{1}{x}$ and $S = [1, \infty)$. We can attain the value 1 but we never reach it!
- $f(x) = x$ and $S = (0, 1)$. The opportunity set is **open**, thus we never attain the value $x = 1$.
- $f(x) = x$ if $x < 1/2$ and $f(x) = x - 1$ if $x \geq 1/2$, and $S = [0, 1]$.

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Bounded Set

Definition

The set S is **bounded** if there exists a number k such that the distance of every point in S from the origin is at most k.

A bounded set does not extend "infinitely" in any direction.

Closed Set

Definition

The set S is **closed** if it contains all its limit points.

We say that a set that is **closed** and **bounded** is **compact**.

Proposition

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- Note that the requirement of boundedness is on the set, not the function!
- Note that the Proposition gives only a **sufficient** condition: **IF** a continuous function is defined in a compact set **THEN** it will definitely have a minimum and a maximum. It does **NOT** rule out the possibility that a function has a minimum and/or maximum if it is not continuous or is not defined in a compact set.

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Necessary conditions for an interior optimum- ONE VARIABLE

We start considering the case of **one** variable.

We already know that if f is differentiable, then there is a relationship between the solution of the optimization problem

$$\max_{\mathbf{x}} f(\mathbf{x}) \text{ subject to } \mathbf{x} \in I$$

and the points in which $f' = 0$.

Call $x : f'(x) = 0$ the **stationary point** of f

Consider the examples below,

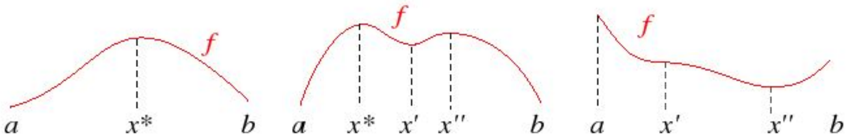

Figure: Stationary points

Proposition

*Let f be a differentiable function of a single variable defined on the interval I. If a point x in the **interior** of I is a local maximizer/minimizer of f, then $f'(x) = 0$.*

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The result gives a **necessary** condition for x to be a maximizer/minimizer of f. We will refer to this condition as the **first order condition** (FOC).

Consider a maximum of a function of **two** variables, x and y. At this point the function must decrease in every direction (along the x-axis and y-axis).

Proposition

Let f be a differentiable function of n variables defined on the compact set S. If the point x in the interior of S is a local maximizer/minimizer of f, then $f_i'(x) = 0 \; \forall i = 1, ..., n$.

...again another **necessary** condition.

Local Optima

Local Optima- ONE VARIABLE

The second-order conditions (SOC) for optimum of function of one variable are summarized as following,

Proposition

Let f a C^1 function defined in the interval I. Suppose x^* is a stationary point of f in the interior of I. Then,

- *If $f''(x^*) < 0$, then x^* is a local maximizer.*
- *If x^* is a local maximizer, then $f''(x^*) \leq 0$.*
- *If $f''(x^*) > 0$, then x^* is a local minimizer.*
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If $f''(x^*) = 0$ then we don't know, without further investigation, whether x^* is a local maximizer or local minimizer of f, or neither (check the functions x^4, $-x^4$, and x^3 at $x = 0$). In this case, information about the signs of the higher order derivatives may tell us whether a point is a local maximum or a local minimum.

Let (x_0, y_0) be a stationary point of the function f of two variables. Suppose it is a local maximizer. Then, we can conclude that

$$f''_{11}(x_0, y_0) \leq 0 \; f''_{22}(x_0, y_0) \leq 0$$

where the f''_{ij} denotes the second partial derivative of f with respect to its ith argument and then to its jth argument.

However, these conditions are not **sufficient** to define a stationary point a local maximizer/minimizer.

The **definiteness** of the Hessian of the function could give the condition we require.

Proposition

Let f be a C^1 function of n variables, defined on the compact set S. Suppose that x^* is a stationary point in the interior of S. Then,

- *If $H(x^*)$ is negative definite then x^* is a local maximizer.*
- *If x^* is a local maximizer then $H(x^*)$ is negative semidefinite.*
- *If $H(x^*)$ is positive definite then x^* is a local minimizer.*
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MANY VARIABLES (cont.)

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Recall that the Hessian of a function f of two variables is

$$H(x^*) = \begin{pmatrix} f''_{11}(x^*) & f''_{12}(x^*) \\ f''_{21}(x^*) & f''_{22}(x^*) \end{pmatrix}$$

The Hessian is ***negative definite*** (or x^* is a local maximizer) if $f''_{11}(x^*) < 0$ and $|H(x^*)| > 0$ (these two inequalities implies that $f''_{22}(x^*) < 0$). In other word,

$$f''_{11}(x^*)f''_{22}(x^*) - f''_{12}(x^*)f''_{21}(x^*) > 0$$

Similarly, for a local minimizer we require the Hessian to be ***positive definite***, or $f''_{11}(x^*) > 0$ and $|H(x^*)| > 0$.

Hessian (cont.)

If $|H(x^*)| < 0$, then x^* is neither a local minimizer nor a local maximizer. In such case x^* is called a **saddle point.**

Global Optima

Conditions for a global optimum-ONE VARIABLE

Let f be a concave differentiable function. Then for every point x, no point on the graph f lies above the tangent to f on x. Thus, if x^* is a stationary point, then x^* is a **global maximizer** of f. More formally,

Proposition

Let f be a differentiable function defined on the interval I, and let x be in the interior of I. Then,

- *If f is concave, then x is a global maximizer of f **if and only if** x is a stationary point of f.*
- *If f is convex, then x is a global minimizer of f **if and only if** x is a stationary point of f.*

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Recall that f is concave if and only if its second derivative is nonpositive (and similarly for a convex function). So we can rewrite the conditions above as,

- $f''(z) \leq 0 \, \forall z \in I \Rightarrow [x$ is a global maximizer of $f \iff f'(x) = 0]$.
- $f''(z) \geq 0 \, \forall z \in I \Rightarrow [x$ is a global minimizer of $f \iff f'(x) = 0]$.

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A differentiable concave function of many variables always lies below, or on, its tangent plane, and a differentiable convex function always lies above, or on, its tangent plane. Therefore, let f be a C^1 function of n variables in a convex set S, and let x be in the interior of S. Then,

- $H(z)$ is negative semidefinite $\forall z \in S \Rightarrow$ [x is a global maximizer of $f \iff f'(x) = 0$].
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Note the different conditions required to have a local maximizer or a global maximizer:

- **Suff. conditions for local maximizer:** x^* is a stationary point of f and the Hessian is negative definite at x^*.
- **Suff. conditions for global maximizer:** x^* is a stationary point of f and the Hessian is negative semidefinite for all values of $x \in S$.

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Optimization:equality constraint

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Two variables - one constraint. Necessary conditions for an optimum

Consider the two-variables problem,

$$\max_{x,y} f(x, y) \text{ subject to } g(x, y) = c$$

The constraint set is a set of points in the (x, y) space such that $g(x, y) = c$. Assume that both the constraint set and the level sets are one-dimensional sets (curves).

Two variables - one constraint. Necessary conditions for an optimum

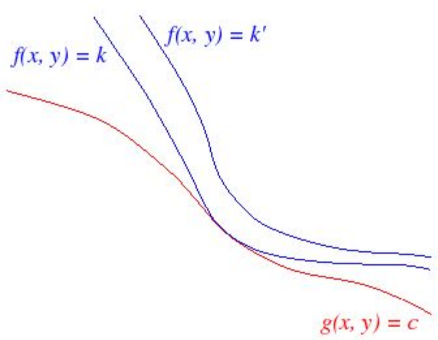

Figure: Optimization problem with equality constraint and two variables

Two variables - one constraint. Necessary conditions for an optimum

Assuming that both f and g are differentiable, we note that the solution (x^*, y^*) is the point where the two curves are tangent, or

$$-\frac{f_1'(x^*, y^*)}{f_2'(x^*, y^*)} = -\frac{g_1'(x^*, y^*)}{g_2'(x^*, y^*)}$$

or

$$\frac{f_1'(x^*, y^*)}{g_1'(x^*, y^*)} = \frac{f_2'(x^*, y^*)}{g_2'(x^*, y^*)}$$

Two variables - one constraint. Necessary conditions for an optimum

Introduce a new variable, λ, equal to,

$$\lambda = \frac{f_1'(x^*, y^*)}{g_1'(x^*, y^*)} = \frac{f_2'(x^*, y^*)}{g_2'(x^*, y^*)}$$

Therefore the conditions for (x^*, y^*) to be an optimum can be rewritten in two equations (+ the constraints that need to be satisfied),

$$\begin{array}{rcl} f_1'(x^*, y^*) - \lambda g_1'(x^*, y^*) & = & 0 \\ f_2'(x^*, y^*) - \lambda g_2'(x^*, y^*) & = & 0 \\ c - g(x^*, y^*) & = & 0 \end{array}$$

Two variables - one constraint. Necessary conditions for an optimum

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f_2'(x^*, y^*) - \lambda g_2'(x^*, y^*) &= 0 \\
c - g(x^*, y^*) &= 0
\end{aligned}
$$

which can be rewritten as the FOC of the **Lagrangean** function

$$\mathcal{L}(x, y) = f(x, y) + \lambda(c - g(x, y))$$

Two variables - one constraint. Necessary conditions for an optimum

Let f and g be two continuous and differentiable functions of two variables defined on the set S. Let c be a number and suppose that (x^*, y^*) is an interior point of S that solves the problem

$$\max_{x,y} f(x,y) \text{ subject to } g(x,y) = c$$

or

$$\min_{x,y} f(x,y) \text{ subject to } g(x,y) = c$$

and suppose that $g_1'(x^*, y^*) \neq 0$ or $g_2'(x^*, y^*) \neq 0$. Then there exists a unique number λ such that (x^*, y^*) is a stationary point of the Lagrangean

$$\mathcal{L}(x,y) = f(x,y) + \lambda(c - g(x,y))$$

Two variables - one constraint. Necessary conditions for an optimum

That is (x^*, y^*) satisfies the FOC

$$\begin{aligned} \mathcal{L}_1' = f_1'(x^*, y^*) - \lambda g_1'(x^*, y^*) &= 0 \\ \mathcal{L}_2' = f_2'(x^*, y^*) - \lambda g_2'(x^*, y^*) &= 0 \end{aligned}$$

in addition to the constraint $c - g(x^*, y^*) = 0$.

Two variables - one constraint. The Lagrangean multiplier

Consider again the problem

$$\max_{x,y} f(x,y) \text{ subject to } g(x,y) = c$$

and suppose we want to solve it for various values of c. Let $(x^*(c), y^*(c))$ be the solution and $\lambda^*(c)$ the Lagrangean multiplier. Assume that the functions $x^*(c), y^*(c), \lambda^*(c)$ are continuous and differentiable, and $g'_1(x^*(c), y^*(c)) \neq 0$ or $g'_2(x^*(c), y^*(c)) \neq 0$.

Let $f^*(c) = f(x^*(c), y^*(c))$. Differentiate $f^*(c)$ with respect to c:

$$\begin{aligned} f'^*(c) &= f'_x(x^*(c), y^*(c))x'^*(c) + f'_y(x^*(c), y^*(c))y'^*(c) \\ &= \lambda'^*(c)\left(g'_x(x^*(c), y^*(c))x'^*(c) + g'_y(x^*(c), y^*(c))y'^*(c)\right) \end{aligned}$$

Two variables - one constraint. The Lagrangean multiplier

But we know that $g(x^*(c), y^*(c)) = c$ for all c. Thus,

$$g'_x(x^*(c), y^*(c))x'^*(c) + g'_y(x^*(c), y^*(c))y'^*(c) = 1 \text{ for all } c$$

Therefore, if we substitute the expression above we obtain,

$$f'^*(c) = \lambda'^*(c)$$

that is **"the value of the Lagrange multiplier at the solution of the problem is equal to the rate of change in the maximal value of the objective function as the constraint is relaxed."**

Two variables - one constraint. Sufficient conditions for an optimum

Again consider the problem

$$\max_{x,y} f(x,y) \text{ subject to } g(x,y) = c$$

and all the assumptions stated above. By substituting for y using the constraint, we can reduce the problem to a one-variable x problem. Let h be implicitly defined by $g(x,h(x)) = c$. Then the problem becomes

$$\max_{x} f(x,h(x)) \text{ subject to } g(x,h(x)) = c$$

Two variables - one constraint. Sufficient conditions for an optimum

Define $F(x) = f(x, h(x))$. Then,

$$F'(x) = f_1'(x, h(x)) + f_2'(x, h(x))h'(x)$$

Let x^* be a stationary point of F. Then, a sufficient condition for x^* to be a local optimum is that $F''(x) < 0$. We have,

$$\begin{aligned} F''(x^*) = \quad & f_1''(x^*, h(x^*)) + 2f_{12}''(x^*, h(x^*))h'(x^*) + \\ & + f_{22}''(x^*, h(x^*))(h'(x^*))^2 + f_2'(x^*, h(x^*))h''(x^*) \end{aligned}$$

Two variables - one constraint. Sufficient conditions for an optimum

Now since $g(x, h(x)) = c$ for all x, we have

$$g_1'(x, h(x)) + g_2'(x, h(x))h'(x) = 0$$

so that

$$h'(x) = -\frac{g_1'(x, h(x))}{g_2'(x, h(x))}$$

Using this expression we can derive $h''(x)$ and substitute into $F''(x^*)$ expression. Rearranging, we obtain

Two variables - one constraint. Sufficient conditions for an optimum

Rearranging, we obtain

$$F''(x^*) = -\frac{D(x^*, y^*, \lambda^*)}{(g_2'(x^*, y^*))^2}$$

where $D(x^*, y^*, \lambda^*)$ is the **Bordered Hessian** of the Lagrangean and defined as

$$\begin{vmatrix} 0 & g_1'(x^*, y^*) & g_2'(x^*, y^*) \\ g_1'(x^*, y^*) & f_{11}''(x^*, y^*) - \lambda^* g_{11}''(x^*, y^*) & f_{12}''(x^*, y^*) - \lambda^* g_{12}''(x^*, y^*) \\ g_2'(x^*, y^*) & f_{21}''(x^*, y^*) - \lambda^* g_{21}''(x^*, y^*) & f_{22}''(x^*, y^*) - \lambda^* g_{22}''(x^*, y^*) \end{vmatrix}$$

Two variables - one constraint. Sufficient conditions for an optimum

Summarizing, consider the problem

$$\max_{x,y} f(x,y) \text{ subject to } g(x,y) = c$$

or

$$\min_{x,y} f(x,y) \text{ subject to } g(x,y) = c$$

and suppose that (x^*, y^*) and λ^* satisfy the FOC

$$\begin{array}{rcl} f_1'(x^*, y^*) - \lambda g_1'(x^*, y^*) & = & 0 \\ f_2'(x^*, y^*) - \lambda g_2'(x^*, y^*) & = & 0 \\ c - g(x^*, y^*) & = & 0 \end{array}$$

Two variables - one constraint. Sufficient conditions for an optimum

Then,

- If $D(x^*, y^*, \lambda^*) > 0$ then (x^*, y^*) is a local maximizer of f subject to constraint $g(x, y) = c$.
- If $D(x^*, y^*, \lambda^*) < 0$ then (x^*, y^*) is a local minimizer of f subject to the constraint $g(x, y) = c$.

Two variables - one constraint. Sufficient conditions for an optimum

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Two variables - one constraint. Sufficient conditions for a global optimum

Let f and g two continuous differentiable functions defined on an open convex set S of two-dimensional space. Suppose that there exists a number λ^* such that (x^*, y^*) is an interior point of S that is a stationary point of the Lagrangean

$$\mathcal{L} = f(x, y) + \lambda(c - g(x, y))$$

Suppose further that $g(x^*, y^*) = c$. Then,

- If $\mathcal{L}$ is **concave**, and in particular, if f is concave and $\lambda^* g$ is convex, then (x^*, y^*) solves the problem $\max_{x,y} f(x, y)$ subject to $g(x, y) = c$.
- If $\mathcal{L}$ is **convex**, and in particular, if f is convex and $\lambda^* g$ is concave, then (x^*, y^*) solves the problem $\min_{x,y} f(x, y)$ subject to $g(x, y) = c$.

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Two variables - one constraint. Sufficient conditions for a global optimum

Note that if g is linear then $\lambda^* g$ is both concave and convex regardless of λ^*. Thus if f is concave and g is linear, any interior point of S that is a stationary point of $\mathcal{L}$ is a solution of the problem.